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HILBERT MODULAR FUNCTIONS AND DIRICHLET SERIES WITH EULER PRODUCTS

Oscar Herrmann

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(Continued) assigned to each form vector. These Dirichlet series possess a functional equation of known structure, and they have a canonical Euler product expansion for form vectors which are eigenvectors of all T-operators.

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HILBERT MODULAR FUNCTIONS AND DIRICHLET SERIES WITH EULER PRODUCTS†

Oscar Herrmann

Blumenthal [1], at the instigation of Hilbert, investigated a generalization of the ordinary modular group which is now known as "Hilbert's Modular Group" on the basis of a totally real algebraic number field. Blumenthal dealt only with questions of algebraic function theory. The modular forms for a number field of arbitrary degree were introduced by Kloosterman, after they had already been applied by Hecke [2], for the special case of real number fields of degree 2, to the solution of number theory problems. An estimation of the order of magnitude of the Fourier coefficients of modular forms is drawn from Kloosterman's investigations [3]. Two works of Maass [4,5] were concerned with generalized groups of the form of Hilbert's modular groups. The major result was to establish the finiteness of the rank of the family of all automorphic forms of a given dimension, given certain constraints on the fundamental domain. /357*

Hecke's [6,7] theory of T-operators was extended by de Bruijn to modular forms of Hilbert's modular group. But since there is not a totally positive generator for every principal ideal, de Bruijn was able to construct a closed operator theory only by composing a modular form from 2^n analytic functions defined in different regions. His operators satisfy a multiplication theorem of the type found in Hecke's theory of rational modular groups. But since de Bruijn's modular forms possess Fourier coefficients only for equivalent

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*Numbers in the margin indicate pagination in the foreign text.

quadratic integer ideals, he was able to define T-operators only for the equivalent quadratic integer ideals. The corresponding Dirichlet series are thus of the form

$$D(s, \lambda) = \sum_{\mathfrak{m}} c(\mathfrak{m})\, \lambda(\mathfrak{m})\, \mathrm{N}(\mathfrak{m})^{-s},$$

summing at most over all equivalent quadratic ideals. An equivalent quadratic ideal is an ideal which differs from an ideal square in at most one principal ideal. Since the unique factorization theorem does not hold in the domain of the equivalent quadratic ideals, such a Dirichlet series possesses no Euler product expansion in the case where the class number h is even. Since the group used by de Bruijn as basis has no discontinuous expansion, there are no Dirichlet series with Euler products for de Bruijn's modular forms in spite of the multiplicative properties of the T-operators. In the present work, vectors are constructed from modular forms in h different groups of the form of Hilbert's modular groups. A theory analogous to that of Hecke and Petersson is developed for these vectors. /358

The groups which are taken as a basis can best be described with the aid of ideal numbers. The definition of the ideal numbers and their most important characteristics from the point of view of applications are collected in §1. At this point, we will state only that the domain of all ideal numbers can be split into h classes $K_1 = K, K_2, \ldots, K_h$, the absolute number classes. In §2, each number class is assigned a system $\Gamma(K_a)$ of matrices

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

The investigation of these matrix systems is extraordinarily complicated because they are in general not even semi-groups. The multiplications permitted in the matrix systems $\Gamma(K_a)$ are described in Lemma 9. For the theory of T-operators, the most important result of this section is Theorem 1, in which systems of representatives are given for the left classes of the matrices $M \in \Gamma(K_a)$ with determinant $\sigma = \alpha\delta - \beta\gamma$ according to the group $\Gamma_1(K_a)$ of all matrices in $\Gamma(K_a)$ with determinant equal to 1. In §3, every matrix M in $\Gamma(K_a)$ is assigned a simultaneous system of linear fractional substitutions. The substitution group $\overline{\Gamma}_1(K_a)$ corresponds to the group $\Gamma_1(K_a)$. Disregarding the finite number of hyperplanes which are the limiting spaces, the group $\overline{\Gamma}_1(K_a)$ is discontinuous in the entire n-dimensional number space in the domain of complex numbers. This space is divided by the limiting spaces into 2^n regions $\hat{T}_\omega$ ($\omega \in K_a$). The union $\hat{T}_{K_a}$ of these regions $\hat{T}_\omega$, that is, the entire space without the limiting spaces, becomes a representation of a system of abstract metric manifolds through the introduction of a metric. The determination of a fundamental domain with simple characteristics for the group $\overline{\Gamma}_1(K_a)$ with respect to $\hat{T}_{K_a}$ follows a method which is a combination of those of Fricke-Klein [9] and Maass [4]. The fundamental domain of the group $\overline{\Gamma}_1(K_a)$ with respect to $\hat{T}_{K_a}$ decomposes into 2^n subdomains; each possesses exactly h apexes, is connected, and is bounded by a finite number of hyperplanes in the direction of the metric (Theorem 4). Since the main emphasis of the investigations lies in the area of function theory, several supplementary observations on the construction of the fundamental domain are left to the reader.

Automorphic forms of dimension $-k$ (k being a natural number), on $\hat{T}_{K_a}$ are observed. Since $\hat{T}_{K_a}$ decomposes into 2^n separate regions, these automorphic forms, the Hilbert modular forms for the class K_a, consist of 2^n separate function branches. De Bruijn has already investigated such functions for the special case of $K_a = K$. A modular form for the class

K_a of dimension $-k$ must be of type $(K_a, -k)$, and $(K_a, -k)$ would denote the family of all modular forms of this type. The family $(K_a, -k)$ is of finite rank (with respect to the domain of the complex numbers), and may be represented as a direct sum of the families $(K_a, -k, \chi)$, where a modular form F of the type $(K_a, -k)$ is a modular form of type $(K_a, -k, \chi)$ whenever $F(\varepsilon\tau) = \chi(\varepsilon)F(\tau)$ is valid for every unity ε and χ is a biquadratic character of the unity group. Not every biquadratic character of the unity group has a corresponding modular form of type $(K, -k, \chi)$. The necessary condition for this is $\chi(\varepsilon^2) = N\varepsilon^{-k}$, and a character χ which satisfies this condition is called an admissible character. The admissible characters are continued to characters of the group $\overset{\times}{Z}$ of all non-zero ideal numbers. The T-operators for modular forms of type $(K_a, -k, \chi)$ are defined by /359

$$T_{K_a}(\sigma) = \mathrm{N}(\sigma)^{k-1}\,\bar{\chi}(\sigma)\sum_{\mu=1}^{f} R_{\mu},$$

where the R_μ's constitute a complete system of representatives for the left classes of matrices in $\Gamma(K_a)$ with determinant σ, according to Theorem 1. The operator $T_{K_a}(\sigma)$ maps the family $(K_a, -k, \chi)$ onto the family $(\sigma, K_a, -k, \chi)$. Thus the iteration of T-operators is possible only when the class displacement is taken into account. Thus the T-operators are in general not commutable. Nevertheless, a multiplication theorem can be proven, and it is in some ways similar to Hecke's [7] multiplication theorem for T-operators for modular groups of higher order.

In order to give a clear overview of the multiplicative properties of the T-operators, we construct the form vectors $\hat{F} = \{F_{K_a}(\tau)\}$ of the type $\{K, -k\}$ each from h forms of type $(K_a, -k)(a = 1, 2,\ldots,h)$. The decomposition of the families $(K_a, -k)$ into the subfamilies $(K_a, -k, \chi)$ is carried over to the family of all form vectors $\hat{F}$ of type $\{K, -k\}$: this family decomposes into the families $\{K, -k, \chi\}$. For these families, the T-operators $T(\sigma)$ are defined by a component-by-component application of $T_{K_a}(\sigma)$ and rearranging so that the a-th component of $\hat{F}|T(\sigma)$ again belongs to the class K_a. The T-operators for the form vectors are commutable. The multiplication formula is especially simple when we limit ourselves to certain subfamilies $\{K, -k, \chi, \psi\}$ of the family $\{K, -k, \chi\}$, where ψ is a character of the number class group. For these subfamilies, the multiplication theorem (Theorem 7) for the T-operators is

$$T(\Sigma)\,T(\sigma) = \sum_{\substack{(\vartheta) \\ 1/\vartheta/(\Sigma,\sigma)}}' T\left(\frac{\Sigma\sigma}{\vartheta^2}\right) \mathrm{N}(\vartheta)^{k-1}\,\psi(\vartheta)\,.$$

The family $\{K, -k,\chi\}$ is the direct sum of the subfamilies $\{K, -k, \chi, \psi\}$. The investigation of the Fourier coefficients of the form vectors leads to a perfect analog to the theory of Hecke and Petersson: The Fourier coefficients $c(\sigma)$ of a form vector of type $\{K, -k, \chi, \psi\}$, which is an eigenvector of all T-operators, conform to the multiplication formula

$$c(\Sigma)\,c(\sigma) = \sum_{\substack{(\vartheta) \\ 1/\vartheta/(\Sigma,\sigma)}}' c\left(\frac{\Sigma\sigma}{\vartheta^2}\right) \mathrm{N}(\vartheta)^{k-1}\,\psi(\vartheta)\,.$$

In particular, the following is thus valid: /360

$$c(\Sigma)\,c(\sigma) = c(\Sigma\sigma) \qquad \text{for} \qquad (\Sigma, \sigma) = 1;$$

that is, $c(\sigma)$ is a multiplicative function (with respect to relatively prime arguments). The inverse is also true: A form vector $\hat{F}$ of type $\{K, -k, \chi, \psi\}$ whose Fourier coefficients are multiplicative is an eigenvector of all T-operators. The family of cusp form vectors of type $\{K, -k, \chi, \psi\}$ is spanned by eigenvectors of all T-operators. There are no further eigenvectors in the family of cusp form vectors. In the subfamily of $\{K, -k, \chi, \psi\}$ which is normal to all cusp forms of this type, there is at least one eigenvector of all T-operators, given that the subfamily is non-empty. Nothing further is known about this subfamily, although one suspects that, when non-empty, it is spanned by exactly one Eisenstein row vector.

The Mellin transformation assigns to each form vector, uniquely and reversibly, a system of Dirichlet series with magnitude characters. These Dirichlet series define integral functions, except for a finite number which have a pole of order 1. They possess a functional equation whose structure is known. By the classical method, and with the aid of the Mellin transformation, this functional equation is reduced to a transformation property of the form vectors. The Dirichlet series

$$D(s, \Lambda) = \sum_{(\sigma)} c(\sigma)\, \Lambda(\sigma)\, \mathrm{N}(\sigma)^{-s}$$

are linearly equivalent with those found by the Mellin transformation. They have a canonical Euler product expansion

$$D(s, \Lambda) = \prod_{(\varrho)} (1 - c(\varrho)\, \Lambda(\varrho)\, \mathrm{N}(\varrho)^{-s} + \psi(\varrho)\, \mathrm{N}(\varrho)^{k-1-2s})^{-1},$$

whenever the form vector $\hat{F}$ is an eigenvector of all T-operations. The product extends over a complete system of non-associated prime numbers ρ. They are the only Euler products which can be found for form vectors of the type $\{K, -k, \chi, \psi\}$.

CHAPTER I
THE GROUP THEORY BASIS

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§1. Ideal Numbers

Let K represent the totally real algebraic number field with (absolute) ideal class number h and degree n which we will use as a basis. Let the n conjugates of K be $K^{(v)}$ $(v = 1, 2,\ldots,n)$. Following Hecke [10] we now construct over K a region Z of ideal numbers with the following properties.

> Z is the smallest region which includes K and in which not only multiplication and division, but also the construction of the greatest common divisor (G.C.D.), are always possible, with the supplementary condition that all unities contained in Z also belong to K.

These conditions define Z unambiguously to the point of isomorphism. We will in some way continue the isomorphs $K \to K^{(v)}$ to isomorphs of Z onto conjugate regions $Z^{(v)}$ $(v = 1, 2,\ldots, n)$. According to this stipulation, every ideal number $\alpha \in Z$ possesses exactly n conjugates $\alpha^{(v)}$ $(v = 1, 2,\ldots,n)$, some of which may of course be identical. Since the regions $Z^{(v)}$ are embedded in the field of complex numbers, the sum and product of ideal numbers with arbitrary complex numbers are defined. The sets of all non-zero ideal numbers in K or Z are denoted by $K^{\times}$ or $Z^{\times}$. The classification of the non-zero ideals into h classes is carried over onto Z: $Z^{\times}$ decomposes into h classes $K^{\times} = K_1^{\times}, K_2^{\times},\ldots,K_n^{\times}$ [Translator's note: Handwritten correction appears in text for last term: $K_h^{\times}$.], the absolute number classes.

Combining these classes $K_a^{\times}$ with the number 0, we get the classes $K_a (a = 1,2,\ldots, h)$ which possess a non-empty intersection. Thus the principle of equivalence which corresponds to this classification of Z

$$\alpha \sim \beta \text{ for } \alpha, \beta \in K_a$$

is not transitive; $\alpha \sim \gamma$ follows from $\alpha \sim \beta$, $\beta \sim \gamma$ if and only if $\beta \neq 0$.

Let the symbol $(\alpha_1, \alpha_2, \ldots, \alpha_r)$ denote an arbitrary G.C.D. of the numbers $\alpha_1, \alpha_2, \ldots, \alpha_r$. The G.C.D. is unambiguously defined down to a unity ε. More specifically, (α) denotes an ideal number which differs from α by at most one unity. An ideal integer is an ideal number which is also totally algebraic. As usual, let α/β [Translator's note: Handwritten correction appears in text: β/α.] denote that $\frac{\alpha}{\beta}$ is an integer.

For later use, we now present several lemmas from the arithmetic of ideal numbers over the field K. Proofs of most of the lemmas can be found in Hecke [10].

Lemma 1: The sum of two non-zero ideal numbers α, β is an ideal number if and only if $\alpha \sim \beta$.

Lemma 2: If $\omega \in Z^{\times}$ and K_a is an arbitrary class of ideal numbers, then there are n ideal numbers α_1, α_2, ..., α_n, such that each ideal number $\alpha \in K_a$ can be expressed unambiguously in the form

$$\alpha = \alpha_1 m_1 + \alpha_2 m_2 + \cdots + \alpha_n m_n$$

with rational coefficients m, and such that ω/α is valid only when all m's are rational integers.

Lemma 3: If $\omega, \alpha \in Z^{\times}$ and if ω/α, then there exists another number γ in the class K_a such that $\omega = (\alpha, \gamma)$. /362

Lemma 4: The G.C.D. ω of two numbers $\alpha, \gamma \in Z^{\times}$ ¢an always be written as a linear combination of α, γ with integer coefficients $\lambda, \mu \in Z$: $\omega = \lambda\alpha + \mu\gamma$.

Furthermore,

$$\lambda \sim \frac{\omega}{\alpha}, \mu \sim \frac{\omega}{\gamma}.$$

A congruence of ideal numbers

$$\alpha \equiv \beta(\mu)$$

with $\mu \neq 0$ indicates that $\frac{\alpha-\beta}{\mu}$ is an integer in Z. Equally, α,β are equivalent and $\frac{\alpha-\beta}{\mu}$ is an integral algebraic number. The following lemmas are true for congruences of ideal numbers:

Lemma 5: A linear congruence

$$\alpha\xi \equiv \beta(\mu)$$

possesses a solution for all $\xi \in Z$ if and only if $(\alpha,\mu)/\beta$. It is mod μ uniquely solvable when $(\alpha,\mu) = 1$ holds. If $\alpha,\beta \in Z^{\times}$, then $\xi \sim \frac{\beta}{\alpha}$.

Lemma 6: A finite system of congruences with pairwise relatively prime modules is simultaneously solvable if each congruence possesses a separate solution such that all separate solutions are equivalent.

Lemma 7: If μ and φ are any integers in $Z^{\times}$, then there exists in every prime residue class mod μ an ideal integer which is relatively prime to φ.

Lemma 8: Let $\mu \in Z^{\times}$ be an integer. Then there exist in every class just as many mod μ incongruent ideal integers. Let this quantity be denoted by $N(\mu)$.

If μ is any ideal number, let $N\mu$ denote the product, $S\mu$, the sum of its n conjugates. If $\mu \in K$, then $N\mu, S\mu$ are rational numbers. Otherwise, $N\mu$ differs from a rational number by at most one root of unity. If μ is an ideal integer, then the quantity $N(\mu)$ introduced in Lemma 8 is equal to $|N\mu|$. Let $\mu \in K$. Then there is a number $\partial \in Z$ (uniquely defined down to some number ε), such that $S\mu$ is an integer if and only if $\partial\mu$ is an integer. This number ∂ is called the differential of K. $d = N(\partial)$ is the discriminant of K.

Up to now, we have been concerned only with the (absolute) partition into classes. Next we observe two finer divisions in $Z^{\times}$.

Two ideal numbers $\alpha, \beta \in Z^{\times}$ are called equivalent in the more restricted sense if $\frac{\alpha}{\beta} \in K^{\times}$ and there exists some number ε such that $\frac{\alpha}{\beta}\varepsilon$ is totally positive. We write $\alpha \approx \beta$. The corresponding number classes are denoted by $\tilde{K}_a$, the class number by $\tilde{h}$. This classification is also applied to the ideals. This is not the case, however, for the most narrow partition which corresponds to the following definition of equivalence: Two ideal numbers $\alpha, \beta \in Z^{\times}$ are equivalent in the <u>most</u> restricted sense when $\alpha \sim \beta$ and $\frac{\alpha}{\beta}$ is totally positive. /363
We write $\alpha \tilde{\approx} \beta$.

Expressions such as σK and $K_a^{\times} K_b^{\times -1}$ are to be interpreted according to the complex calculation familiar from group theory. Thus K_a^{-1} is not defined, as $0 \in K_a$ has no inverse. We therefore define K_a^{-1} as the set resulting from the joining of 0 to $K_a^{\times -1}$. Then it is clear that $K_a^{-1} = K\, K_a^{\times -1}$.

§2. The Arithmetic of Two-row Matrices with Ideal Numbers

In this section, we investigate two-row matrices $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \sigma \end{pmatrix}$ whose elements are ideal integers which fulfill the conditions

$$\alpha \in K,\ \beta \in K_a^{-1} K_b,\ \gamma \in K_a,\ \delta \in K_b \tag{1}$$

and whose determinant is

$$\sigma = \alpha\delta - \beta\gamma \neq 0 \tag{2}$$

The set of all matrices M which satisfy these conditions for fixed K_a and arbitrary K_b we denote by $\Gamma(K_a)$. If all elements of a matrix M are non-zero, then (1) becomes

$$\alpha \in K, \frac{\gamma}{\alpha}, \frac{\delta}{\beta} \in K_a. \tag{3}$$

Condition (3) seems more natural than (1), but it is not useful when an element of the matrix M disappears. It is immediately obvious that $\sigma \in K_b$ is valid. The matrices $M \in \Gamma(K_a)$ do not form a group, and, where $h \neq 1$, not even a semigroup. All that we can say is:

Lemma 9: If $M \in \Gamma(K_a)$ and σ is the determinant of M, then $MM' \in \Gamma(K_a)$ where M' is an arbitrary matrix from $\Gamma(\sigma^{-1}K_a)$.

The proof is obvious when one observes that the number classes are groups with respect to addition. Products other than those named in Lemma 9 do not generally represent matrices with elements from Z and are thus not treated.

We observe that all matrices $L = \begin{pmatrix} \lambda & \mu \\ \nu & \rho \end{pmatrix} \in \Gamma(K_a)$, whose determinant $\lambda\rho - \mu\nu$ is equal to 1 form a group $\Gamma_1(K_a)$. We obtain an overview of all groups in $\Gamma(K_a)$ which comprise $\Gamma_1(K_a)$. The unity groups appear in the process. As a general convention, let $\langle \varepsilon_1 \rangle$ denote a unity group in K whenever

ε_1 represents an arbitrary element of the group. Next let $\Gamma_{\varepsilon_1}(K_a)$ be the set of all matrices $L \in \Gamma(K_a)$ whose determinant is in $\langle\varepsilon_1\rangle$. The sets $\Gamma_{\varepsilon_1}(K_a)$ are groups: the only groups in $\Gamma(K_a)$ which comprise $\Gamma_1(K_a)$. In fact, if one starts with an arbitrary group $\Gamma \in \Gamma(K_a)$ and assigns to each matrix L its determinant as an image, then the resulting projection is a homomorphism of the matrix group onto an integer multiplicative group. But this is a unity group $\langle\varepsilon_1\rangle$. Thus there is at least one $L \in \Gamma$ with determinant ε_1 for every ε_1. If in /364
addition $\Gamma_1(K_a) \in \Gamma$, then all matrices with determinant ε_1 are clearly included in the group Γ_1; that is, $\Gamma = \Gamma_{\varepsilon_1}(K_a)$. From now on let ε denote an arbitrary, ε_0 a totally positive arbitrary unity in K_1.

Lemma 10: If $\Gamma_{\varepsilon_1}(K_a)$, $\Gamma_{\varepsilon_2}(K_a)$ are two matrix groups with

$$\Gamma_1(K_a) \subset \Gamma_{\varepsilon_1}(K_a) \subset \Gamma_{\varepsilon_2}(K_a),$$

then $\Gamma_{\varepsilon_1}(K_a)$ is the self-conjugate subgroup of $\Gamma_{\varepsilon_2}(K_a)$ and the following isomorphism is valid:

$$\frac{\Gamma_{\varepsilon_2}(K_a)}{\Gamma_{\varepsilon_1}(K_a)} \cong \frac{\langle\varepsilon_2\rangle}{\langle\varepsilon_1\rangle}.$$

The proof follows immediately from the second law of isomorphism in group theory when one observes that the kernel of the homomorphism $L \to \varepsilon$ is the group $\Gamma_1(K_a)$.

From Lemma 10, with ε and ε_0 in the designated roles, we derive:

Lemma 11: The groups $\Gamma_{\varepsilon^2}(K_a)$ and $\Gamma_{\varepsilon_0^2}(K_a)$ are self-conjugate subgroups of $\Gamma_{\varepsilon}(K_a)$, of finite index and possessing an abelian factor group. The following is valid:

$$(\Gamma_{\varepsilon}(K_a) : \Gamma_{\varepsilon^2}(K_a)) = 2^n,\ (\Gamma_{\varepsilon}(K_a) : \Gamma_{\varepsilon_0^2}(K_a)) = 2^g$$

where $n \leqq g \leqq 2n-1$. The factor groups are of the form

$$(2,\ldots,2) \text{ or } (2,\ldots,2,4,\ldots,4).$$

The proof is clear when one considers the corresponding unity groups, for which the theorem is known to be valid.

Let a matrix $M \in \Gamma(K_a)$ whose determinant differs from σ only by a unit be called of order (σ). In particular, the matrix $L = \begin{pmatrix} \lambda & \mu \\ \nu & \rho \end{pmatrix}$, with determinant of an arbitrary unity, has order (1). In the future, we will use the notation $L = \begin{pmatrix} \lambda & \mu \\ \nu & \rho \end{pmatrix}$ only for matrices of order (1). Matrices of order (1) are called unimodular matrices. If the determinant of the matrix L is actually 1, then L is properly unimodular.

Multiplying the matrices $M \in \Gamma(K_a)$ of order (σ) from the left by matrices $\in \Gamma_1(K_a)$, and from the right by matrices $\in \Gamma(\sigma^{-1}K_a)$, yields product matrices which, according to Lemma 9, are also matrices from $\Gamma(K_a)$ of order (σ). This leads to the following definition:

Two matrices $M, M' \in \Gamma(K_a)$ of order (σ) are called

left-equivalent or right-equivalent

when there is a matrix

$$L \in \Gamma_1(K_a) \qquad \text{or} \qquad L \in \Gamma_1(\sigma^{-1}K_a)$$

such that

$$LM = M' \qquad \text{or} \qquad ML = M'.$$

We write

$$M \underset{l}{\sim} M' \qquad \text{or} \qquad M \underset{r}{\sim} M'.$$

Since the unimodular substitutions of each class form a group, these equivalence relations are reflexive, symmetrical, and transitive. Thus they motivate a partition into classes. /365
For the definition of the T-operators, we need a complete description of the left partition; the right partition is less important.

Lemma 12: If α, α', γ, γ' are ideal numbers with

$$\begin{matrix} \alpha, \alpha' \in K, \\ \gamma, \gamma' \in K_a, \end{matrix} \qquad (\alpha, \gamma) = (\alpha' \gamma') \neq (0),$$

then there exists an $L \in \Gamma_1(K_a)$ such that

$$\begin{pmatrix} \alpha' \\ \gamma' \end{pmatrix} = L \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \tag{4}$$

Proof: According to Lemma 4, $\omega = (\alpha, \gamma)$ is a linear combination of α, γ with linear coefficients; thus this is also valid for every multiple of ω, such as α' and γ'. Thus there exist integer ideal numbers λ_1, μ_1, ν_1, ρ_1 such that

$$\begin{aligned} \alpha' &= \lambda_1 \alpha + \mu_1 \gamma \\ \gamma' &= \nu_1 \alpha + \varrho_1 \gamma \end{aligned}$$

and such that λ_1, $\rho_1 \in K$, $\nu_1 \in K_a$, $\mu_1 \in K_a^{-1}$. Set

$$\begin{aligned} \lambda &= \lambda_1 + \gamma \xi, \qquad \mu = \mu_1 - \alpha \xi, \\ \nu &= \nu_1 - \gamma \eta, \qquad \varrho = \varrho_1 + \alpha \eta \end{aligned}$$

where $\xi \in K_a^{-1}$, $\eta \in K$. Choosing the proper ξ, η we can assure that

$$\eta \alpha' + \xi \gamma' = 1 - (\lambda_1 \varrho_1 - \mu_1 \nu_1)$$

and that λ, μ, ν, ρ are integers. But then $L = \begin{pmatrix} \lambda & \mu \\ \nu & \rho \end{pmatrix} \in \Gamma_1(K_a)$, and (4) is satisfied.

If η, κ, ρ are integers from $Z^\times$ and if K_a is an arbitrary number class, then the matrix $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma(K_a)$ is called "of the form $\{\eta, \kappa, \sigma, K_a\}$" if the following is true [Translator's note: The following is handwritten beside 1.: $\eta = (\alpha\gamma)$, $K = (\beta, \delta)$] :

$$1.\ \eta / (\alpha, \gamma), \qquad \kappa / (\beta, \delta),$$
$$2.\ \sigma = \alpha\delta - \beta\gamma, \quad (\sigma) = \eta\kappa$$

We write

$$M \in \{\eta, \kappa, \sigma, K_a\}.$$

A fixed matrix of the form $\{\eta, \kappa, \sigma, K_a\}$ whose choice is not critical is denoted by $R(\eta, \kappa, \sigma, K_a)$.

This definition leads to:

Lemma 13: If $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is a matrix of the form $\{\eta, \kappa, \sigma, K_a\}$, then

$$\eta = (\alpha, \gamma) \text{ and } \kappa = (\beta, \delta).$$

Proof: We know that $\sigma = \alpha\delta - \beta\gamma$; thus

$$(\alpha, \gamma)(\beta, \delta) / (\alpha\delta, \beta\gamma) / (\alpha\delta - \beta\gamma) = (\sigma).$$

This leads to

$$\frac{(\alpha, \gamma)}{\eta} \cdot \frac{(\beta, \delta)}{\kappa} \Big/ \frac{(\sigma)}{\eta\kappa} = 1$$

from which the proof is clear.

Lemma 14: If η, κ, σ are integers from $Z^\times$ with /366
$\eta\kappa = (\sigma)$, and if K_a is an arbitrary number class, then

there exists a matrix of the form $\{\eta, \kappa, \sigma, K_a\}$.

Proof: According to Lemma 3, there exist two numbers $\alpha \in K$, $\gamma \in K_a$, such that $\eta = (\alpha, \gamma)$. And, from Lemma 4, there exist two integers λ, μ such that $\eta = \lambda\alpha + \mu\gamma$. With $\delta = \kappa\lambda$, $\beta = -\kappa\mu$, we get:

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \{\eta, \kappa, \sigma, K_a\}.$$

Lemma 15: If M is a matrix of the form $\{\eta, \kappa, \sigma, K_a\}$ then a matrix $M' \in \Gamma(K_a)$ is left-equivalent to M if and only if M' is of the same form.

Proof: 1. Let $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \{\eta, \kappa, \sigma, K_a\}$, $L = \begin{pmatrix} \lambda & \mu \\ \nu & \varrho \end{pmatrix} \in \Gamma_1(K_a)$ and let

$$M' = L\,M = \begin{pmatrix} \lambda\alpha + \mu\gamma & \lambda\beta + \mu\delta \\ \nu\alpha + \varrho\gamma & \nu\beta + \varrho\delta \end{pmatrix} \in \Gamma(K_a).$$

Then clearly $M' \in \{\eta, \kappa, \sigma, K_a\}$.

2. Conversely, let M and M' be placed in $\{\eta, \kappa, \sigma, K_a\}$. Then, from the definition of $\Gamma_1(K_a)$, we have:

$$L = M' M^{-1} = \begin{pmatrix} \dfrac{\alpha'\delta - \beta'\gamma}{\sigma} & \dfrac{\beta'\alpha - \alpha'\beta}{\sigma} \\ \dfrac{\gamma'\delta - \delta'\gamma}{\sigma} & \dfrac{\delta'\alpha - \gamma'\beta}{\sigma} \end{pmatrix} \in \Gamma_1(K_a).$$

Thus $M' \overset{\sim}{1} M$, and Lemma 15 is proven.

Lemma 16: Let $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma(K_a)$ and $\sigma = \alpha\delta - \beta\gamma$. Then there exist two ideal integers $\eta, \kappa \in Z^{\times}$, one ideal number $\zeta \in \sigma K_a^{-1}$ with $\frac{1}{\eta}/\zeta$, and a corresponding matrix $M_0 \in \{\eta, \kappa, \sigma, K_a\}$, such that $M = M_0 U^{\zeta}$ where $U^{\zeta} = \begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix}$. For a given M, $\{\eta, \kappa, \sigma, K_a\}$ is uniquely determined and ζ mod $\frac{\kappa}{\eta}$ is uniquely determined.

Proof: Let $\eta = (\alpha,\gamma)$. We will choose a number $\gamma' \in K_a^\times$ with σ/γ'. According to Lemma 3, there exists a number $\alpha' \in K$ such that $\eta = (\alpha', \gamma')$. According to Lemma 12, there exists a matrix $L \in \Gamma_1(K_a)$ such that

$$\begin{pmatrix}\alpha'\\ \gamma'\end{pmatrix} = L\begin{pmatrix}\alpha\\ \gamma\end{pmatrix}$$

Thus the matrix

$$M' = \begin{pmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{pmatrix} = L\,M$$

satisfies the conditions σ/γ' $\eta = (\alpha', \gamma')$. $\sigma/\alpha'\delta'$ follows from σ/γ'. But α' and σ have a G.C.D. η, because otherwise we would have $(\alpha', \gamma') \neq \eta$. Thus $\kappa = \frac{\sigma}{\eta}/\delta'$. We choose a number $\zeta \in \sigma^{-1} K_a^{-1}$ with

$$\zeta \equiv 0\left(\frac{1}{\eta}\right), \quad \alpha'\zeta \equiv \beta'(\kappa). \tag{5}$$

This is possible because the congruences

$$\xi \equiv 0\,(\alpha\,\eta^{-1}),\ \xi \equiv \beta'(\kappa) \tag{6}$$

are individually solvable with $\xi \in \sigma K^{-1}$ and have relatively prime modules. According to Lemma 6, the congruences in (6) are thus simultaneously solvable. If ξ is a solution of (6), then $\zeta = \frac{\xi}{\alpha}$ is a solution of (5). Then clearly /367

$$M'' = M'\,U^{-\zeta} = \begin{pmatrix}\alpha' & \beta'-\alpha'\zeta\\ \gamma' & \delta'-\gamma'\zeta\end{pmatrix} \in \{\eta, \kappa, \sigma, K_a\}.$$

We have $M = L^{-1}M''U^{\zeta}$, and according to Lemma 15 $M_0 = L^{-1}M'' \in \{\eta, \kappa, \sigma, K_a\}$. Since the elements in the first column and the determinant are not affected when right multiplying by U^{ζ}, $\{\eta, \kappa, \sigma, K_a\}$ is uniquely determined.

All that remains is the proof that $\zeta \bmod \frac{\kappa}{\eta}$ is uniquely determined. It is sufficient to prove that:

If $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and $M' = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$ are two matrices from the form $\{\eta, \kappa, \sigma, K_a\}$ and if

$$M' = M\, U^{\zeta}, \tag{7}$$

then $\frac{\kappa}{\eta} / \zeta$ is valid. But this is clear because (7) yields:

$$\begin{aligned} \alpha' &= \alpha, & \beta' &= \beta + \alpha\,\zeta, \\ \gamma' &= \gamma, & \delta' &= \delta + \gamma\,\zeta, \end{aligned}$$

and thus $\kappa/(\alpha\zeta, \gamma\zeta) = \eta\zeta$, and the lemma is proven.

Theorem 1: If σ is an integer from $Z^{\times}$, then a complete system of representatives for the left classes of matrices $M \in \Gamma(K_a)$ with determinant σ is given by:

$$R(\eta, \kappa, \sigma, K_a)\, U^{\zeta},$$

where (η) traverses a complete system of non-associated divisors of σ, and where κ is determined by $\eta\kappa = \sigma$. $\zeta \in \sigma\, K_a^{-1}$ traverses a complete system of mod $\frac{\kappa}{\eta}$ differing multiples of $\frac{1}{\rho}$ [Translator's note: Handwritten correction appears in text for last term: $\frac{1}{\eta}$] for fixed η, κ. The number of these left classes is given by

$$\sum_{(\eta)} N(\eta).$$

The first part of the theorem follows directly from the lemmas already proven. The second part follows from simple enumeration.

Definition: Two matrices $M, M' \in \Gamma(K_a)$ of order (σ) are called absolutely equivalent if there exist matrices $L \in \Gamma_1(K_a)$ and $L' \in \Gamma_1(\sigma^{-1} K_a)$ such that

$$M' = L\, M\, L'$$

The notation used is $M' \approx M$.

Theorem 2: Two matrices $M, M' \in \Gamma(K_a)$ of order (σ) are equivalent if and only if they have the same determinant and the same G.C.D. The G.C.D. of a matrix is understood to be the G.C.D. of its elements.

Proof: 1. It is clear that equivalent matrices have identical determinants and identical G.C.D.'s

2. To prove the inverse, it suffices to show that an /368
arbitrary matrix M with determinant σ and G.C.D. ϑ is equivalent to a previously determined matrix $R\left(\frac{\sigma}{\vartheta}, \vartheta, \sigma, K_a\right)$. Since the G.C.D. of the matrix M is not always within the domain, we cannot confine ourselves to primitive matrices without making the condition in (1) more general.

Let $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and $\vartheta = (\alpha, \beta, \gamma, \delta)$. We choose numbers $\gamma' \in K_a$ with σ/γ' and $\alpha' \in K$ where $(\alpha, \gamma) = (\alpha', \gamma')$. There exists a corresponding matrix $L_1 \in \Gamma_1(K_a)$ such that

$$M' = L_1 M = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}$$

We set

$$M'' = U^\lambda M' U^\nu = \begin{pmatrix} \alpha'' & \beta'' \\ \gamma' & \delta'' \end{pmatrix} = \begin{pmatrix} \alpha' + \lambda\gamma' & \beta' + \lambda\delta' + \nu\alpha' + \lambda\nu\gamma' \\ \gamma' & \delta' + \nu\gamma' \end{pmatrix}$$

with integral λ, ν where $\lambda \in K_a^{-1}$ and $\nu \in \sigma K_a^{-1}$ (which we will have occasion to use again later) and $\mu = (\alpha', \delta')$. It follows that

$$(\mu, \beta'') = (\alpha'', \delta'', \beta'') = (\alpha', \beta', \sigma, \gamma'),$$

because σ is a linear combination of α', β' with integral coefficients. We also know that

$$\vartheta / (\mu, \beta') = (\alpha', \beta', \sigma, \delta') / (\alpha', \beta', \gamma', \delta') = \vartheta .$$

and thus

$$\vartheta = (\mu, \beta').$$

The values

$$\xi = \beta' + \lambda\,\delta' + \nu\,\alpha'$$

traverse all numbers which are congruent mod μ to β' when λ, ν independently traverse all integers of those classes. There exist numbers λ, ν such that $\vartheta = (\xi, \gamma')$. This can be proved by dividing the two conditions

$$\xi \equiv \beta'\ (\mu),\ (\xi, \gamma') = \vartheta$$

by ϑ and applying Lemma 6. But then

$$(\beta'', \gamma') = (\xi + \lambda\,\nu\,\gamma', \gamma') = (\xi, \gamma') = \vartheta.$$

It follows that

$$\left(\beta'', \frac{\sigma}{\vartheta}\right) = (\beta'', \delta') = \vartheta.$$

In fact, we have

$$\vartheta \Big/ \left(\beta'', \frac{\sigma}{\vartheta}\right) \Big/ (\beta'', \gamma') = \vartheta$$

and

$$\vartheta/(\beta'', \delta'') = \left(\beta'', \delta'', \frac{\sigma}{\vartheta}\right) \Big/ \left(\beta'', \frac{\sigma}{\vartheta}\right) = \vartheta.$$

This yields the solvability of the congruence

$$\beta''\,\eta \equiv \alpha \left(\frac{\sigma}{\vartheta}\right)$$

for integer $\eta \in \sigma^{-1}K_a$, that is, there exists a $\eta \in \sigma^{-1}K_a$ such /369 that

$$M''' = \begin{pmatrix} \alpha''' & \beta''' \\ \gamma''' & \delta''' \end{pmatrix} = \begin{pmatrix} \alpha'' - \eta\,\beta'' & \beta'' \\ \gamma' - \eta\,\delta'' & \delta'' \end{pmatrix} = M'' \begin{pmatrix} 1 & 0 \\ -\eta & 1 \end{pmatrix}$$

satisfies the condition $\frac{\sigma}{\vartheta}/\alpha'''$. This yields $\sigma/\beta''\gamma'''$, and, finally, $\frac{\sigma}{\vartheta}/\gamma''' = \vartheta$, because $\left(\frac{\sigma}{\vartheta}, \beta''\right)$. Thus $M''' \in \{\frac{\sigma}{\vartheta}, \vartheta, \sigma, K_a\}$. Since all matrices of this form are left-equivalent, M''' can be transformed into the representative R by left-multiplying with a properly chosen matrix L_2. An application of this general theorem yields:

Lemma 17: Let R_0 be a fixed matrix of the form $\{\sigma, \sigma, \sigma^2, \sigma K_a\}$. There exists a corresponding system of representatives

$$R_1, R_2, \ldots, R_j \qquad \text{where} \qquad j = \sum_{(\vartheta)} N(\vartheta)$$

of the left classes of matrices from $\Gamma(K_a)$ with determinant σ such that the matrices

$$R_0 R_1^{-1}, R_0 R_2^{-1}, \ldots, R_0 R_j^{-1}$$

constitute a complete system of representatives of matrices from $\Gamma(K_a)$ with determinant σ.

Proof: Let the matrices $R'_\mu \in \Gamma(K_a)$ and $R''_\mu \in \Gamma(\sigma K_a)$ be two systems of representatives of matrices with determinant σ. We can assume without restrictions that two matrices with the same index possess the same G.C.D. Thus the matrices $R_0 R'^{-1}_\mu$, R''_μ have pairwise the same G.C.D. According to Theorem 2, they are equivalent, that is, there exist two matrices L'_μ, L''_μ for each μ such that

$$R_0 R'^{-1}_\mu = L''_\mu R''_\mu L'_\mu$$

Thus the matrices $L''_\mu R''_\mu$ constitute a complete system of representatives of the left classes of matrices with determinant σ of the sort required by Lemma 17.

Of course this lemma can also be formulated as follows:

Lemma 17a: There exists a common system of representatives of the left and right classes of matrices $M \in \Gamma(K_a)$ with determinant σ.

§3. Substitutions and Substitution Groups

We now leave the domain K to investigate its n conjugates $K^{(1)}, K^{(2)}, \ldots, K^{(n)}$, which, according to our established conditions, are all real. We assign to each domain $K^{(v)}$ a complex variable $\tau^{(v)}$ $(v = 1, 2, \ldots, n)$. These variables are the conjugates of τ. Because the regions $Z^{(v)}$ are embedded in the domain of the complex numbers, multiplication of ideal numbers with arbitrary complex numbers is defined. We now establish the convention that an equation or an inequality which includes numbers from K and the variable τ represent a shorthand for the system of conjugate equations or inequalities.

This means, for instance, that the substitution /370

$$\tau \to \tau' = M\tau = \frac{\alpha\tau + \beta}{\gamma\tau + \delta}, \tag{8}$$

where $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ denotes a matrix with elements from Z which is analogous to the system

$$\tau^{(v)} \to \tau'^{(v)} = M^{(v)}\tau^{(v)} = \frac{\alpha^{(v)}\tau^{(v)} + \beta^{(v)}}{\gamma^{(v)}\tau^{(v)} + \delta^{(v)}}. \tag{9}$$

The matrix $M^{(v)} = \begin{pmatrix} \alpha^{(v)} & \beta^{(v)} \\ \gamma^{(v)} & \delta^{(v)} \end{pmatrix}$ is called the v-th conjugate of M. For $\alpha \in K$, the inequality $\alpha > 0$ is identical to the system $\alpha > 0$ $(v = 1, 2, \ldots, n)$; that is, $\alpha > 0$ means that α is totally positive. $\alpha \in K$ was stipulated to insure that all conjugates at α would be real.

We compute $\operatorname{Im} \omega'\tau'$ for $M \in \Gamma(K_a)$ and $\omega' \in K_a^{\times}$:

$$\begin{aligned} \operatorname{Im} \omega'\tau' &= \operatorname{Im} \omega' \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \\ &= |\gamma\tau + \delta|^{-2} \operatorname{Im} \omega'(\alpha\tau + \beta)(\bar{\gamma}\bar{\tau} + \bar{\delta}). \end{aligned}$$

This can be written as

$$\operatorname{Im} \omega' \tau' = |\gamma \tau + \delta|^{-2} \operatorname{Im} \omega' (\bar\alpha \bar\delta - \bar\beta \bar\gamma) \tau$$
$$= |\gamma \tau + \delta|^{-2} |\sigma|^2 \operatorname{Im} \omega' \sigma^{-1} \tau, \ldots$$

since $\alpha, \omega'\bar\alpha\bar\gamma\tau\bar\tau$, $\omega'\beta\bar\delta$ are real and $\operatorname{Im} \omega'\beta\overline{\gamma\tau} = -\operatorname{Im} \overline{\omega'}\,\overline{\beta}\gamma\tau =$
$= -\operatorname{Im} \omega'\beta\overline{\gamma}\tau$.

Setting $\omega'\sigma^{-1} = \omega$, we get:

$$\operatorname{Im} \omega' \tau' = |\gamma \tau + \delta|^{-2} |\sigma|^2 \operatorname{Im} \omega \tau. \tag{10}$$

The region $\hat{T}_\omega$ is now defined by

$$\operatorname{Im} \omega \tau > 0$$

where ω is an arbitrary ideal number from $Z^\times$. Clearly, $\hat{T}_\omega$ is dependent in the most restricted sense only on the class of ω.

Lemma 18: If $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma(K_a)$ and $\sigma = \alpha\delta - \beta\gamma$, then the substitution $\tau \to \tau' = M\tau$ maps the region $\hat{T}_\omega$ onto the region $\hat{T}_{\omega'}$, where $\omega' \in K_a^\times$ and $\omega \approx \sigma^{-1}\omega'$. The following are true:

$$\operatorname{Im} \frac{\omega'\tau'}{|\omega'|} = |\gamma \tau + \delta|^{-2} |\sigma| \operatorname{Im} \frac{\omega\tau}{|\omega|}, \tag{11}$$
$$\operatorname{Im} \frac{\omega\tau}{|\omega|} = |\gamma \tau' - \alpha|^{-2} |\sigma| \operatorname{Im} \frac{\omega'\tau'}{|\omega'|}, \tag{12}$$
$$\frac{\omega \bar\omega \, d\tau \, d\bar\tau}{(\operatorname{Im} \omega\tau)^2} = \frac{\omega' \omega' \, d\tau' \, d\bar\tau'}{(\operatorname{Im} \omega'\tau')^2}. \tag{13}$$

Proof: If we set ω' equal to $\omega\sigma$, (11) is equivalent to (10). But since the left side of (11) is dependent in the most restricted sense only on the number class of ω', (11) is universally valid. (12) and (13) are likewise proved by direct calculation checks, first setting $\omega' = \sigma\omega$, and then later removing this restriction.

We set $\hat{T}_{K_a} = \bigcup_{\omega \in k_a} \hat{T}_\omega$. We regard every individual $\hat{T}_\omega$ as /371
a representation of an abstract manifold, so that $\hat{T}_{K_a}$ represents a system of abstract manifolds. Through the parameter τ,

these abstract manifolds are projected onto 2^n disjoint regions of the n-dimensional complex number space, so that the entire number space is covered, aside from a finite number of hyper planes, the limiting spaces.

The system $\hat{T}_{K_a}$ becomes a system of metric manifolds when we define a metric in $\hat{T}_\omega$ through the quadratic form

$$(d_\omega s)^2 = S\,\frac{\omega\overline{\omega}\,d\tau\,d\overline{\tau}}{(\operatorname{Im}\omega\tau)^2}$$

Clearly, this metric is dependent only on $\hat{T}_\omega$. Let the geodesic distance between two points τ, $\tau^* \in \hat{T}_\omega$ be denoted by $E_\omega(\tau,\tau^*)$. The following lemma should be obvious:

Lemma 19: Given $M \in \Gamma(K_a)$, (8) maps the system $\hat{T}_{\sigma^{-1}K_a}$ onto the system $\hat{T}_{K_a}$ such that $\hat{T}_\omega \subset \hat{T}_{\sigma^{-1}K_a}$ becomes $\hat{T}_{\sigma\omega} \subset \hat{T}_{K_a}$ with preservation of length.

If we set

$z = x + iy = \frac{\omega\tau}{|\omega|}$, that is, $z^{(v)} = x^{(v)} + iy^{(v)} = \frac{\omega^{(v)}\tau^{(v)}}{|\omega^{(v)}|}$

then $\hat{T}_\omega$ is mapped onto the region $y > 0$, and the metric fundamental form becomes

$$ds^2 = S\,\frac{dx^2 + dy^2}{y^2}.$$

Thus the metric already investigated by Maass lies in the region $y > 0$. From reference [4] we take the distance formula

$$E_\omega\left(\tau, \frac{|\omega| i}{\omega}\right) = \sqrt{S\left(\log\frac{\left|\frac{\omega\tau}{|\omega|}+i\right| + \left|\frac{\omega\tau}{|\omega|}-i\right|}{\left|\frac{\omega\tau}{|\omega|}+i\right| - \left|\frac{\omega\tau}{|\omega|}-i\right|}\right)^2}. \qquad (14)$$

A substitution (8) is assigned to each matrix M. The set of all substitutions assigned to the matrices $M \in \Gamma(K_a)$ is denoted by $\overline{\Gamma}(K_a)$; similarly, the substitution group $\overline{\Gamma}_{\varepsilon_1}(K_a)$ is assigned to the matrix group $\overline{\Gamma}_{\varepsilon_1}(K_a)$. We require a fundamental domain for the group $\overline{\Gamma}_1(K_a)$ with respect to $\hat{T}_{K_\alpha}$. The determination of a fundamental domain $\hat{P}_1(K_a)$ for $\overline{\Gamma}_1(K_a)$ is most easily accomplished with the aid of a metrization according to a procedure already developed by Fricke and Klein [9]. But a direct application of this procedure yields no information about the convergence of the fundamental domain towards the limiting spaces. Maass [4] gives a method which yields a fundamental domain with more useful characteristics. A significant simplication is achieved in our case through a combination of the two methods.

In each $\hat{T}_\omega$, we distinguish a point

$$\tau_\omega = \frac{|\omega|\, i\, y_0}{\omega}, \; y_0 > 1$$

where all conjugates of y_0 are identical and independent of ω. /372

Lemma 20: Let $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \{\vartheta, \vartheta, \sigma, K_a\}$, then either $M\tau_\omega = \pm \tau_\omega$ or $M\tau_\omega \neq \tau_{\omega'}$ for all $\tau_{\omega'}$.

Proof: Because of equation (12), it follows from τ_ω that

$$y_0 = \operatorname{Im} \frac{\omega \tau_\omega}{|\omega|} = |\gamma\, \tau_{\omega'} - \alpha|^{-2}\, |\sigma| \; \operatorname{Im} \frac{\omega' \tau_{\omega'}}{|\omega'|} = |\gamma\, \tau_{\omega'} - \alpha|^{-2}\, |\sigma|\, y_0 .$$

and thus

$$|\sigma| = |\gamma\, \tau_{\omega'} - \alpha|^2 .$$

Because $\operatorname{Im} (\gamma\tau_{\omega'} - \alpha) = \operatorname{Im} \gamma\tau_{\omega'} = \gamma \frac{|\omega'|}{|\omega|} y_0$ it follows that

$$|\sigma| \geqq \left| \gamma \frac{|\omega'|}{\omega'}\, y_0 \right|^2 = |\gamma|^2\, y_0^2 > |\gamma|^2 ,$$

for $\gamma \neq 0$. This leads to a contradiction with σ/γ^2, thus $\gamma = 0$. Thus $(\alpha) = (\delta) = (\vartheta)$. Then M is of the form $\begin{pmatrix} \varepsilon\delta & \delta\zeta \\ 0 & \delta \end{pmatrix}$, where ε is a unity and $\zeta \in K_a^{-1}$ is an integer. Thus we have

$$\tau_{\omega'} = \varepsilon\tau_\omega + \zeta \quad \text{with} \quad \omega' = \varepsilon\,\omega\,.$$

Multiplying this equation by $\omega' \in K_a$ yields $\mathrm{Re}\,\omega'\zeta = 0$; additionally, $\omega'\zeta$ is an element of K_a, and is thus real. This proves $\zeta = 0$. It follows immediately from $\tau_\omega' = \varepsilon\tau_\omega$ that $|\varepsilon| = 1$, thus $\varepsilon = \pm 1$, and thus $\tau_{\omega'} = \pm\tau_\omega$.

We define the regions $\hat{P}_{\omega'} \subset \hat{T}_{\omega'}$ by the following inequalities:

$$E_{\omega'}(\tau', \tau_{\omega'}) \leqq E_{\omega'}(\tau', M\,\tau_\omega) = E_\omega(M^{-1}\tau, \tau_\omega), \tag{15}$$

where $M \in \Gamma(K_a)$ traverses all matrices of the form $\{\vartheta, \vartheta, \sigma, K_a\}$, with variables $\vartheta, \sigma, \omega$ and $\omega = \sigma^{-1}\omega'$. When y_0 is chosen large enough, the region $\hat{P}_{\omega'}$ satisfies the inequalities

$$|x'| \leqq C_1 \tag{16}$$

$$\frac{1}{C_2} \leqq \frac{y'}{\sqrt{\mathrm{N}\,y'}} \leqq C_2 \quad \text{with} \quad \begin{cases} x' = \mathrm{Re}\,\dfrac{\omega'\tau'}{|\omega'|^2} \\ y' = \mathrm{Im}\,\dfrac{\omega'\tau'}{|\omega'|}\,. \end{cases} \tag{17}$$

$$y' \geqq c_1 \tag{18}$$

$C_1, C_2, \ldots$ and $c_1, c_2, \ldots$ shall be taken to mean positive constants dependent only on the domain K. The C's should be chosen sufficiently large, the c's sufficiently small. In particular, we shall always assume $C_a > 1$, $c_a < 1$.

The first two inequalities are easy to prove when one observes that $\hat{P}_{\omega'}$ is contained in the region $\hat{P}^*_{\omega'}$ ($\subset \hat{T}_{\omega'}$), which is defined by the system of inequalities arising from (15)

when we let M traverse only matrices of the form $\begin{pmatrix} 1 & \beta_{-1} \\ 0 & \varepsilon_0 \end{pmatrix}$.
Clearly $\hat{P}^*_{\omega'}$ is a fundamental domain of the group of all transformations $\tau \to \varepsilon_0(\tau + \beta)$ with respect to $\hat{T}_{\omega'}$. The goal of the following lemmas is to prove the third inequality.

Lemma 21: There exists only one constant c_2 which is dependent on domain K and such that for every $\tau' \in \hat{T}_{\omega'}$ there are two ideal integers $\alpha \in K$, $\gamma \in K_a$ where /373

$$y' \geqq c_2 |\gamma \tau' - \alpha|^2 \tag{19}$$

can be found.

Proof: The inequalities

$$\left|\gamma \frac{|\omega'|}{\omega'} x' - \alpha\right| \leqq \frac{1}{\sqrt{2 c_2}} \sqrt{y'};\ |\gamma| \leqq \frac{1}{\sqrt{y'}} \frac{1}{\sqrt{2 c_2}} \tag{20}$$

are solvable for α, $\gamma \neq 0$, 0 if c_2 is small enough, as can be proven by the introduction of a base for the field K and the class K_a according to Lemma 2, and the application of a theorem of linear forms of Minkowski. The inequalities (19) follow immediately from (20).

Lemma 22: The column $\begin{pmatrix} \alpha \\ \gamma \end{pmatrix}$ assigned to τ' by Lemma 21 can be made into a matrix $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \{\vartheta, \vartheta, \sigma, K_a\}$ such that $\tau = M^{-1}\tau' \in \hat{P}_{\omega}$, where $\vartheta = (\alpha, \gamma)$.

Proof: As is apparent from the proof of Lemma 14, the column $\begin{pmatrix} \alpha \\ \gamma \end{pmatrix}$ can be made into a matrix $M' = \begin{pmatrix} \alpha & \beta' \\ \gamma & \delta' \end{pmatrix} \in \{\vartheta, \vartheta, \vartheta^2, K_a\}$. This matrix can be right-multiplied by an arbitrary matrix $L = \begin{pmatrix} 1 & \beta_{-1} \\ 0 & \varepsilon_0 \end{pmatrix}$ without changing the first column. We set $M = M'L$. Then $M \in \{\vartheta, \vartheta, \vartheta^2, \varepsilon_0^{-1}, K_a\}$. The proper choice of L will always insure that $\tau = L^{-1}M'^{-1}\tau' \in \hat{P}^*_{\omega}$, since $\hat{P}^*_{\omega}$ is the fundamental domain of all L's of the given form. Lemma 22 is thus proven with $\sigma = \vartheta^2 \varepsilon_0^{-1}$.

In addition, there exist constants C_4 which are dependent only on K and not on y_0, such that for every point $\tau \in \hat{P}^*_\omega$ the following inequalities hold:

$$\frac{1}{\sqrt{n}} S \left|\log \frac{y}{y_0}\right| - C_4 \leqq E_\omega(\tau, \tau_\omega) \leqq \frac{1}{\sqrt{n}} S \left|\log \frac{y}{y_0}\right| + C_4 \tag{21}$$

We get this by first joining τ to τ_ω along a properly chosen path consisting of two geodetic parts and then applying the triangle inequality to the component parts. The length of the components is found from (14).

To prove (18), we choose a constant $y_0 > e^{1/\sqrt{2}}$ Let τ' be an element of $\hat{P}^*_\omega$, with $y' = \operatorname{Im} \dfrac{\omega'\tau'}{|\omega'|} < c_2$. According to Lemma 22 we choose the matrix M such that $M \in \{\vartheta, \vartheta, \sigma, K_a\}$, $\tau = M^{-1}\tau' \in \hat{P}^*_\omega$, $y > c_2$. Then either $\sqrt[n]{y} > c_2^{-1} y_0$ or $c_2 \leq \sqrt[n]{N y} \leq c_2^{-1} y_2$

In the first case, we get $\tau' \in \hat{P}_{\omega'}$. This is so because it follows from $\sqrt[n]{Ny} > c_2^{-1} y_0$, $y' < c_2$ that $\gamma \neq 0$ and $Nyy' < 1$ (compare the proof of Lemma 20). This yields

$$\log N y_0^2 > 2\sqrt{n}\, C_4 > 2\sqrt{n}\, C_4 + \log N y y'$$

and

$$\frac{1}{\sqrt{n}} S \left|\log \frac{y'}{y_0}\right| - C_4 > \frac{1}{\sqrt{n}} S \left|\log \frac{y}{y_0}\right| + C_4.$$

It follows from (21) that /374

$$E_{\omega'}(\tau', \tau_{\omega'}) \geqq \frac{1}{\sqrt{n}} S \left|\log \frac{y'}{y_0}\right| - C_4 > \frac{1}{\sqrt{n}} S \left|\log \frac{y}{y_0}\right| + C_4 \geqq E_\omega(\tau, \tau_\omega).$$

In the second case, τ is located in a compactum, the intersection of $\hat{P}^*_\omega$ with the region $c_2 \leq \sqrt[n]{Ny} \leqq c_2^{-1} y_0$. Thus we have $E_\omega(\tau, \tau_\omega) < C_5$, with the constant C_5 dependent only on K.

But we know that $E_{\omega'}(\tau', \tau_{\omega'}) > C_5$ when $y' < c_3$ for a sufficiently small positive constant $c_3 \leqq c_2$. Looking at (17), we see that (18) is proven for the proper choice of the constant $c_1 \leqq c_3$.

By similar reasoring, we can prove that the region $\hat{P}_{\omega'}$ contains the path $\tau = i\frac{|\omega'|}{\omega'} t$, where $y_0 \leqq t \leqq \infty$ and t is the same for all conjugates. This means that the point $\infty = (\infty^{(1)}, \infty^{(2)}, \ldots, \infty^{(n)})$ is a boundary point of $\hat{P}_{\omega'}$.

Observing that the points $M\tau_{\omega}$ can be obtained from a finite number of such points by applying all substitutions of the discrete group Γ_1 (K_a), we see that the points $M\tau_{\omega}$ do not cluster in any finite closed region $\hat{B}$. This means that, except for a neighborhood of the cusp (∞), the region $\hat{P}_{\omega'}$ is bounded only by a finite number of hyperplanes in the sense of the hyberbolic metric. In a sufficiently small neighborhood of the cusp (∞), however, the boundary of $\hat{P}_{\omega'}$ is contained in the boundary of $\hat{P}^*_{\omega'}$. Clearly, $\hat{P}^*_{\omega'}$ possesses only a finite number of boundary hyperplanes, and thus $\hat{P}_{\omega'}$ is also bounded by a finite number of hyperplane portions. In addition, since $\hat{P}_{\omega'}$ is the intersection of hyperbolic half-spaces, it is a hyperbolically convex region. Since $\hat{P}_{\omega'}$ is bounded by finitely many hyperbolic hyperplanes, and is contained in the region defined by the inequalities $|x| \leqq C_1$, $y \geqq c_1$, $\hat{P}_{\omega}$ is measurable and has a finite hyperbolic volume. Thus we have proved.

Theorem 3: The region $\hat{P}_{\omega'}$ is a convex region with finite volume and a (parabolic) cusp, and is bounded by a finite number of hyperplane portions.

Take an integer ϑ from each number class K_b^* ($b=1,2,\ldots,h$). and let ϑ represent the G.C.D. of a number $\alpha \in K$ and a number $\gamma \in K_a$. Extend the column $\begin{pmatrix}\alpha\\ \gamma\end{pmatrix}$ to a matrix $M = \begin{pmatrix}\alpha & \beta\\ \gamma & \zeta\end{pmatrix} \in \{\vartheta, \vartheta, \sigma, K_a\}$, where σ traverses a complete modulo system of the group $\langle \varepsilon\rangle$ of non-identical numbers which differ from ϑ^2 by at most some number ε. Then construct the finite number of regions $MP_{\sigma^{-1}\omega'}$. From Lemma 20, their union of sets is a fundamental domain of the group

$\bar{\Gamma}_1(K_a)$ with respect to $T_{\omega'}$. Substituting equivalent regions for the $M\hat{P}_{\sigma^{-1}\omega'}$ in the familiar manner, we can assure that the fundamental domain is connected. This proves:

Theorem 4: The group $\hat{\Gamma}_1(K_a)$ possesses with respect to $\hat{T}_{K_a}$ a fundamental domain $\hat{P}_1(K_a)$ consisting of 2^n connected regions. $\hat{P}_1(K_a)$ is bounded by a finite number of hyperbolic cusps and has a finite, non-Euclidian volume.

In like manner, we can construct fundamental domains $\hat{P}_{\varepsilon_i}(K_a)$, /375 for the groups $\bar{\Gamma}_{\varepsilon_i}(K_a)$ from the regions $\hat{P}_\omega$ and their images.

CHAPTER II

FORMS AND FORM VECTORS

§4. Modular Forms for the Class K

Let $F(\tau) = F(\tau^{(1)}, \tau^{(2)}, \ldots, \tau^{(n)})$ be an analytic function of the n complex variables $\tau^{(1)}, \tau^{(2)}, \ldots, \tau^{(n)}$ defined in some region. Then for every matrix $M = \begin{pmatrix}\alpha & \beta \\ \gamma & \zeta\end{pmatrix}$, there is a corresponding operator

$$F(\tau) \mid M = \mathrm{N}(\gamma \tau + \delta)^{-k} F(M \tau), \tag{22}$$

It is clear how the domain of definition of the function $F(\tau)$ must be transformed when applying operators such as (22), which are called basic operators. For two basic operators M, M^1, the product and the linear combination with complex coefficients λ, λ' are defined by

$$F(\tau) \mid (M \cdot M') = (F(\tau) \mid M) \mid M' \tag{23}$$

$$F(\tau) \mid (\lambda M + \lambda' M') = \lambda F(\tau) \mid M + \lambda' F(\tau) \mid M'. \tag{24}$$

It is easy to see that operator multiplication is isomorphic to matrix multiplication. When using (24) we must insure that the domain of definition of $F(\tau)|M$ agrees with that of $F(\tau)\ M'$.

The modular forms for the class K_a of dimension $-k$ (k, a natural number) are characterized through the following four conditions (compare [3]):

1. $F(\tau)$ is defined and regular in $\hat{T}_{K_a}$.
2. $F(\tau)$ satisfies the transformation formula
$$F(\tau) \mid L = F(\tau) \text{ for } L \in \Gamma_1(K_a).$$
3. $F(\tau)|M$ is limited to $y > 1$ for all $M \in \Gamma(K_a)$,

assuming $y = \mathrm{Im}\,\frac{\omega\tau}{|\omega|}$ where $\omega \in \sigma^{-1}K_a$ and $\sigma = \alpha\delta - \beta\gamma$.

4. If $F(\tau)$ satisfies the supplementary condition

$$\lim_{y \to \infty} F(\tau) \mid M = 0,$$

assuming $\tau = \frac{|\omega| iy}{\omega}$, $y^{(v)} = y$ ($v = 1, 2, \ldots, n$), then let $F(\tau)$ be called a cusp form.

A modular form for the class K_a of dimension $-k$ will be called of type $(K_a, -k)$; if $F(\tau)$ is a cusp form of type $(K_a, -k)$, we will call $F(\tau)$ of type $(K_a, -k)_s$. We write $F(\tau) \in (K_a, -k)$ on $F(\tau) \in (K_a, -k)_s$.

The domain of definition of the modular forms for the class K_a decomposes into 2^n separate regions $\hat{T}_{\omega'}$ ($\omega' \in K_a$). This means that a modular form for the class K_a always consists of 2^n analytical function branches which have nothing whatever to do with each other. If we take the function branches of all modular forms defined in the region $\hat{T}_{\omega'}$ to be functions of $z = \omega' \tau / \omega'$, we get /376 a system of automorphic forms such as those investigated by Maass [5]. From reference [5], we know that these function branches in the region $\hat{T}_{\omega'}$ constitute a linear family of finite rank. But this fact implies that the modular forms according to our definition also constitute a linear family of finite rank, since our family of all modular forms can be written as a direct sum of form families in the cited work. Thus we get

Theorem 5: The modular forms of type $(K_a, -k)$ constitute a linear family of finite rank.

Lemma 23: Let $M, M' \in \Gamma(K_a)$, $M \,\tilde{\updownarrow}\, M'$ and $F(\tau) \in (K_a, -k)$; then $F(\tau)|M = F(\tau)|M'$.

Proof: We have stipulated that $M' = LM$ with $L \in \Gamma_1(K_a)$; thus it follows that

$$F \mid M' = F \mid L\,M = (F \mid L) \mid M = F \mid M.$$

Lemma 24: Let $L^* \in \Gamma_\varepsilon(K_a)$, $F \in (K_a, -k)$; then it is also true that $F^* = F|L^* \in (K_a, -k)$. If F is a cusp form, then so is F^*.

Proof: 1. The substitution L^* carries $\hat{T}_{K_a}$ into itself, thus F^* is defined in $\hat{T}_{K_a}$.

2. Let $L \in \Gamma_1(K_a)$. Then $L^* \tilde{1}\ L^*L$, since $\Gamma_1(K_a)$ is a self-conjugate subgroup in $\Gamma_\varepsilon(K_a)$. Thus, according to Lemma 21,

$$F^* \mid L = F \mid L^* L \doteq F \mid L^* = F^*.$$

3 and 4. Just as for M_1 it is true that $M^* = L^*M \in \Gamma(K_a)$ and that

$$F^* \mid M = F \mid L^* M = F \mid M^*.$$

It follows from this that F^* also satisfies conditions 3 and 4.

Lemma 25: If $L \in \Gamma_{\varepsilon^4}(K_a)$, that is, if the determinant of $L \in \Gamma(K_a)$ is a fourth power of some unity ε, then $F|L = F$ for every modular form of type $(K_a, -k)$.

Proof: Every $L \in \Gamma_{\varepsilon^4}(K_a)$ can be written in the form

$$L = L_1 \begin{pmatrix} \varepsilon^2 & 0 \\ 0 & \varepsilon^2 \end{pmatrix} \text{ with } L_1 \in \Gamma_1(K_a).$$

Thus, according to Lemma 23,

$$F(\tau) \mid L = F(\tau) \mid \begin{pmatrix} \varepsilon^2 & 0 \\ 0 & \varepsilon^2 \end{pmatrix} = N\,\varepsilon^{-2k} F(\tau).$$

Because $N\varepsilon = \pm 1$, it is true that $N\varepsilon^{-2k} = 1$, and Lemma 25 follows.

Now let χ be a biquadratic character of the unity group. Just as for every modular form $F(\tau) \in (K_a, -k)$, it is then also true that

$$F_\chi(\tau) = \frac{1}{(\varepsilon : \varepsilon^4)} \sum_{\varepsilon \bmod \langle \varepsilon^4 \rangle} \bar{\chi}(\varepsilon) F(\varepsilon\tau) \in (K_a, -k);$$

because, according to Lemma 24, $F(\varepsilon\tau) = F(\tau) \Big| \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix}$ is a module form of type $(K_a, -k)$; thus the same is true for even linear combination of finitely many $F(\varepsilon\tau)$. According to Lemma 25, $F_\chi(\tau)$ is independent of the choice of a system of representatives mod $\langle \varepsilon^4 \rangle$. In addition, /377

$$F_\chi(\varepsilon_1 \tau) = \chi(\varepsilon_1) F_\chi(\tau)$$

is still valid for every unity ε_1. In fact,

$$F_\chi(\varepsilon_1 \tau) = \frac{1}{(\varepsilon : \varepsilon^4)} \sum_\varepsilon \bar{\chi}(\varepsilon) F(\varepsilon \varepsilon_1 \tau) = \chi(\varepsilon_1) \frac{1}{(\varepsilon : \varepsilon^4)} \sum_\varepsilon \bar{\chi}(\varepsilon\varepsilon_1) F(\varepsilon\varepsilon_1\tau) = \chi(\varepsilon_1) F_\chi(\tau).$$

Inversely, it follows on the basis of known relations for abelian characters that

$$F(\tau) = \sum_\chi F_\chi(\tau),$$

summing over all biquadratic characters.

A modular form of type $(K_a, -k)$ will be called of character χ, or of the type $(K_a, -k, \chi)$, if it satisfies the extra condition

5. $F(\varepsilon\tau) = \chi(\varepsilon)F(\tau)$ for all $\varepsilon \in K$, where χ is a biquadratic character of the unity group.

Let a cusp form of type $(K_a, -k, \chi)$ be called of type

$(K_a, -k, \chi)_g$. Thus $(K_a, -k, \chi)_g = (K_a, -k) \cap (K_a, -k, \chi)$. Since the intersection of the families $(K_a, -k, \chi)$ and $(K_a, -k, \chi_1)$, for the case $\chi \neq \chi_1$ clearly includes only the null form, the family of all modular forms of type $(K_a, -k)$ is the direct sum of the families $(K_a, -k, \chi)$. Thus in particular, the families $(K_a, -k, \chi)$ and $(K_a, -k, \chi)$ possess a finite rank.

The following lemma follows directly from condition 2 and condition 5 for forms of type $(K_a, -k, \chi)$:

Lemma 26: If $L \in \Gamma_\varepsilon(K_a)$ and $F(\tau) \quad (K_a, -k, \chi)$, then

$$F(\tau) \mid L = \chi(\varepsilon_1) F(\tau),$$

where ε_1 is the determinant of L.

We cannot necessarily find modular forms which differ from the null form for every biquadratic character of the unity group. For instance, setting $L = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}$, the definition of the basic operator yields $F(\tau) \mid L = N\varepsilon^{-k} F(\tau)$ and Lemma 26 yields $F(\tau) \mid L = \chi_\varepsilon{}^2 F(\tau)$. It follows that

$$\chi(\varepsilon^2) = N\,\varepsilon^{-k}, \tag{25}$$

provided that $F(\tau)$ is not the null form. Characters which satisfy this condition are called admissable characters. In particular, setting $\varepsilon = -1$ in Eq. (25) yields

$$1 = \chi(1) = \chi((-1)^2) = (N-1)^k = (-1)^{nk}.$$

Thus if nk is odd, no admissable character χ exists. Then there would be no modular forms of type $(K_a, -k, \chi)$ (χ arbitrary) which differ from the null form. It follows from the above that every modular form of type $(K_a, -k)$ vanishes identically.

From now on, we will always stipulate that nk be even and that χ be an admissable character. For later applications, we continue the character χ to a character of the group $Z^{\times}$ of all non-zero ideal numbers σ. For this, we choose a complete system of prime numbers ρ and define /378

$$\chi(\varrho) = \sqrt{N(\varrho)^k N \varrho^{-k}},$$

where the root must somehow be determined. Since the unique factorization theorem is valid in $Z^{\times}$, $\chi(\sigma)$ is defined for all numbers $\sigma \in Z^{\times}$ when we set $\chi(\sigma) = \chi(\varepsilon) \Pi (\chi(\rho))^{\nu\rho}$ for $\sigma = \varepsilon \Pi \rho^{\nu\rho}$ and when it is true that

$$\chi(\sigma)\,\chi(\sigma') = \chi(\sigma\sigma').$$

In addition, we have

$$\chi(\sigma^2) = N(\sigma)^k N \sigma^{-k}. \tag{26}$$

This equation is a generalization of the admissability condition, and will be necessary later.

Since $F(\tau)|M$ is dependent only on the left class M, $R(\eta, \chi, \sigma, K_a)$ as an operator for modular forms of type $(K_a, -k)$ is uniquely determined. Every left-equivalency relation for matrices now has a corresponding equation of operators. Equations of operators will be labeled by the addition of the phrase "for $(K_a, -k)$" or "for $(K_a, -k, \chi)$." We now list some equations of operators for later use.

$$R(\eta, \kappa, \sigma, K_a) = R(\eta, \eta, \eta^2, K_a) \begin{pmatrix} 1 & 0 \\ 0 & \eta^{-2}\sigma \end{pmatrix} \quad \text{for} \quad (K_a, -k) \tag{27}$$

$$R(\vartheta, \vartheta, \vartheta^2, K_a)\, R(\eta, \varkappa, \sigma, \vartheta^{-2} K_a = R(\vartheta\,\eta, \vartheta\varkappa, \vartheta^2\sigma, K_a). \qquad (28)$$

Indeed, right multiplying a matrix of the form $(\eta, \eta, \eta^2, K_a)$ by $\begin{pmatrix} 1 & 0 \\ 0 & \eta^2\sigma \end{pmatrix}$ yields a product matrix which, according to Lemma 9, is contained in $\Gamma(K_a)$ and has determinant σ, and for which the G.C.D. of the first column is unaltered, while the G.C.D. of the second column is $\eta\frac{\sigma}{\eta^2} = \frac{\sigma}{\eta} = \kappa$ (28) can be proven in analogous fashion. Lemma 26 can be written as an equation of operators for the family $(K_a, -k, \chi)$:

$$R(1, 1, \varepsilon, K_a) = \chi(\varepsilon) \qquad \text{for} \qquad (K_a, -k, \chi)$$

From

$$R(1, 1, \varepsilon, K_a)\, R(\eta, \kappa\, \sigma, K_a) = R(\eta, \kappa, \sigma\varepsilon, K_a)$$

we get the equation of operators

$$\chi(\varepsilon)\, R(\eta, \kappa, \sigma, K_a) = R(\eta, \kappa, \varepsilon, \sigma K_a) \qquad \text{for} \qquad (K_a, -k, \chi) \qquad (29)$$

In the following lemma, we use upper- and lower-case Greek letters in like manner, in order to avoid superscripts.

Lemma 27: Let $\begin{pmatrix} A & B \\ \Gamma & \Delta \end{pmatrix} \in \{H, K, \Sigma, K_a\}$, $\begin{pmatrix} \alpha & \beta \\ \gamma & \sigma \end{pmatrix} \in \{\eta, \kappa, \sigma, \Sigma^{-1} K_a\}$ and $\Sigma\sigma/(\Gamma, \gamma)$. Let $\frac{1}{H}/Z$ and $Z \in \Sigma^{-1} K_a$. Then there exists a number $\lambda \in \Sigma^{-1}\sigma^{-1} K_a$ such that

$$\frac{K}{H\eta}\Big/\lambda, \; \lambda\alpha \equiv \beta + Z\delta\left(\frac{K\kappa}{H}\right) \qquad (30)$$

If Z traverses a full system of mod $\frac{K}{H}$ different multiples of /379

$\frac{1}{H}$ for fixed H, then λ traverses a full system of mod $\frac{K\kappa}{H\eta}$ different multiples of $\frac{K}{H\eta}$. In addition,

$$\begin{pmatrix} A & B \\ \Gamma & \Delta \end{pmatrix} U^z \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} U^{-\gamma} \in \{H\eta, K\kappa, \Sigma\sigma, K_a\}. \tag{31}$$

Proof: (30) is mod $\frac{K\kappa}{H\eta}$ uniquely solvable, since setting $\lambda = \lambda^* \frac{K}{H\eta}$ yields a congruence mod K which according to Lemma 5 is uniquely solvable with integer λ^*. Inversely, for a given $\lambda \in \Sigma^{-1}\sigma^{-1} K_a$, we can regard $\frac{K}{H\eta}/\lambda$ as a congruence for $Z \in \Sigma^{-1}K_a$ with $\frac{1}{H}/Z$. Again, Lemma 5 shows that this congruence is solvable. The proof of (31) is now clear.

§5. The T-operators for modular forms of the class K_a

In this section, F shall always be taken to mean a modular form of type $(K_a, -k, \chi)$. Equations of operators are always regarded as equations of operators for modular forms of this type, so we will drop the phrase "for $(K_a, -k, \chi)$" for the sake of simplicity.

Utilizing a complete system of representatives for the left classes of all matrices with determinant σ constructed according to Theorem 1, we define, following de Bruijn [8], the operator $T_{K_a}(\sigma)$ for all modular forms of type $(K_a, -k, \chi)$:

$$T_{K_a}(\sigma) = \mathrm{N}(\sigma)^{k-1}\,\bar{\chi}(\sigma) \sum_{\substack{(\eta),\zeta \\ 1/\eta/\sigma,\ \eta\kappa=\sigma \\ \frac{1}{\eta}/\zeta\in\sigma^{-1}K_a,\ \zeta \bmod \frac{\kappa}{\eta}}} R(\eta,\kappa,\sigma,K)\,U^{\zeta}.$$

It is immediately obvious that $T_{K_a}(\sigma)$ is a linear operator. We also have the following lemma:

Lemma 28: The operator $T_{K_a}(\sigma)$ as operator for the family $(K_a, -k, \chi)$ is dependent only on (σ).

Proof: That the T-operator is independent of the choice of the system of representatives of the left classes for fixed σ follows from Lemma 23. The invariance of the operator when σ is replaced by $\varepsilon\sigma$ follows from Eq. (29).

Lemma 29: Let $F(\tau) \in (K_a, -k, \chi)$; then $F(\tau) | T_{K_a}(\sigma) \in (\sigma^{-1}K_a, -k, \chi)$. If $F(\tau)$ is a cusp form, so is $F(\tau) | T_{K_a}(\sigma)$.

Proof: Verify the defining conditions for the modular forms of type $(\sigma^{-1}K_a, -k, \chi)$ are $(\sigma^{-1}K_a, -k, \chi)_s$ one by one (compare the proof of Lemma 24).

Thus the T-operators can be applied repeatedly one after another, taking the class indices into account, of course. Thus the T-operators are in general not commutable. Instead, we have the following theorem: /380

Theorem 6: For modular forms of type $(K_a, -k, \chi)$, the following operator equation* holds:

$$T_{K_a}(\Sigma)\cdot T_{\Sigma^{-1}K_a}(\sigma) = \sum_{\substack{(\vartheta)\\ 1/\vartheta/(\Sigma,\sigma)}} R(\vartheta,\vartheta,\vartheta^2,K_a)\,\mathrm{N}(\vartheta)^{2k-1}\,\chi(\vartheta^2)\,T_{\vartheta^2 K_a}$$

Proof: From the definition, we have

$$\begin{aligned} &T_{K_a}(\Sigma)\,T_{\Sigma^{-1}K_a}(\sigma)\,\mathrm{N}(\Sigma\,\sigma)^{k-1}\,\chi(\Sigma\,\sigma)\\ &=\sum_{(\mathrm{H}),\mathrm{Z}} R(\mathrm{H},\mathrm{K},\Sigma,K_a)\,U^{\mathrm{Z}}\sum_{(\eta),\zeta} R(\eta,\kappa,\sigma,\Sigma^{-1}K_a)\,U^{\zeta}\\ &=\sum_{(\mathrm{H}),(\eta),\mathrm{Z},\zeta} R(\mathrm{H},\mathrm{K},\Sigma,K_a)\,U^{\mathrm{Z}}\,R(\eta,\kappa,\sigma,\Sigma^{-1}K_a)\,U^{\zeta}, \end{aligned} \qquad (32)$$

where the conditions of summation are to be taken from the defining equation of the T-operators. Since the choice of representatives is not critical, we can assume that they have been chosen such that Lemma 27 applies. So the right side of (32) can be changed to

$$=\sum_{\mathrm{H},\eta}\sum_{\mathrm{Z},\zeta} R(\mathrm{H}\,\eta,\mathrm{K}\kappa,\Sigma\sigma,K_a)\,U^{\lambda+\zeta}$$

where λ is a function of Z for fixed H, η. According to Lemma 27, we can now introduce λ as a new variable of summation in place of Z. We get

$$=\sum_{\substack{(\mathrm{H}),(\eta)\\ 1/\mathrm{H}/\Sigma\\ 1/\eta/\sigma}}\ \sum_{\substack{\lambda,\kappa\\ \frac{\kappa}{\mathrm{H}\eta}/\lambda\in\Sigma^{-1}\sigma^{-1}K_a,\ \lambda \bmod \frac{\mathrm{K}\kappa}{\mathrm{H}\eta}\\ \frac{1}{\eta}/\zeta\in\Sigma^{-1}\sigma^{-1}K_a,\ \zeta \bmod \frac{\kappa}{\eta}}} R(\mathrm{H}\,\eta,\mathrm{K}\kappa,\Sigma\,\sigma,K_a)\,U^{\lambda+\zeta}$$

A further transformation of the variables of summation is begun by

$$\vartheta=(\mathrm{K},\mathrm{H}),\quad \mu=\frac{\mathrm{H}\,\eta}{(\mathrm{K},\mathrm{H})} \qquad (33)$$

*[The following term is handwritten in original at the end of the equation: $\left(\frac{\Sigma\,\sigma}{\Theta^2}\right)$.]

It is now always true that

$$1/\vartheta/(\Sigma, \sigma) \text{ and } 1/\mu/\Sigma\,\sigma.$$

Inversely, for every pair ϑ, μ which satisfies these conditions, there exists a pair H, η determined down to unities, such that (33) holds, namely

$$\eta = \frac{(\sigma, \vartheta\,\mu)}{\vartheta}, \quad H = \frac{\vartheta^2\,\mu}{(\sigma, \vartheta\,\mu)}. \tag{34}$$

Furthermore, according to Lemma 16, $\xi = \lambda + \zeta$ is only critical mod $K\chi/H_\eta$. For fixed H, η, however, there are exactly $N(\vartheta)$ pairs λ, ζ to which the same ξ mod $K\chi/H_\eta$ must be assigned. Applying these substitutions, the right side of (32) finally becomes

$$= \sum_{(\vartheta),(\mu),\xi} R\left(\vartheta\,\mu, \frac{\Sigma\,\sigma}{\vartheta\,\mu}, \Sigma\,\sigma, K_a\right) \mathrm{N}(\vartheta)\,U^{\xi}.$$

$\vartheta \,/\, \frac{\Sigma\sigma}{\vartheta\mu}$ is valid, thus (28) can be applied, yielding

$$= \sum_{(\vartheta),(\mu),\xi} R(\vartheta, \vartheta, \vartheta^2, K_a)\, R\left(\mu, \frac{\Sigma\,\sigma}{\vartheta^2\,\mu}, \frac{\Sigma\,\sigma}{\vartheta^2}\,\vartheta^{-2}\,K_a\right) \mathrm{N}(\vartheta)\,U^{\xi}.$$

Thus /381

$$\begin{aligned} T_{K_a}(\Sigma)\,T_{\Sigma^{-1}K_a}(\sigma) &= \sum_{(\vartheta),(\mu),\xi} \mathrm{N}(\Sigma\,\sigma)^{k-1}\,\bar{\chi}(\Sigma\,\sigma)\,\mathrm{N}(\vartheta) \\ &\quad \times R(\vartheta, \vartheta, \vartheta^2, K_a)\, R\left(\mu, \frac{\Sigma\,\sigma}{\vartheta^2\,\mu}, \frac{\Sigma\,\sigma}{\vartheta^2}, \vartheta^{-2}\,K_a\right) U^{\xi} \\ &= \sum_{\substack{(\vartheta) \\ 1/\vartheta/(\Sigma,\sigma)}} \mathrm{N}(\vartheta)^{2k-1}\,\chi(\vartheta^2)\, R(\vartheta, \vartheta, \vartheta^2\,K_a)\, \bar{\chi}\left(\frac{\Sigma\,\sigma}{\vartheta^2}\right) \mathrm{N}\left(\frac{\Sigma\,\sigma}{\vartheta^2}\right)^{k-1} \\ &\quad \times \sum_{\substack{(\mu),\xi \\ 1/\mu/\frac{\Sigma\,\sigma}{\vartheta^2} \\ \frac{1}{\mu}/\xi\in\Sigma^{-1}\sigma^{-1}K_a \\ \xi \bmod \frac{\Sigma\,\sigma}{\vartheta^2\,\mu^2}}} R\left(\mu, \frac{\Sigma\,\sigma}{\vartheta^2\,\mu}, \frac{\Sigma\,\sigma}{\vartheta^2}, \vartheta^2\,K_a\right) U^{\xi}. \end{aligned}$$

The multiplication theorem follows immediately. For the case h = 1, we see that the T-operators are also commutable.

§6. Form Vectors

We construct a vector

$$\hat{F} = \{F_{K_a}(\tau)\}.$$

from h modular forms $F_{K_a}(\tau) \in (K_a, -k)$ $(a = 1, 2, \ldots, h)$.

We call such a vector a form vector of type $\{K, -k\}$. All form vectors of type $\{K, -k\}$ constitute a linear family which is clearly the direct sum of the families $(K_a, -k)$, and is thus of finite rank. We call the h modular forms $F_{K_a}(\tau)$ $(a = 1, 2, \ldots, h)$ the components of $\hat{F}$. When all components of $\hat{F}$ are cusp forms, we call it a cusp form vector, or of type $\{K, -k\}_s$; when all components $F_{K_a}(\tau)$ $(a = 1,2,\ldots,h)$ are of character χ, we call F of type $\{K, -k, \chi\}$. The type $\{K, -k, \chi\}_s$ is defined analogously.

Since the components of a form vector of type $\{K, -k\}$ can be split up by characters, the family of all form vectors of type $\{K, -k\}$ is the direct sum of all families of the form $\{K, -k, \chi\}$.

We now define the operators $R(\eta, \kappa, \sigma)$ for form vectors of type $\{K, -k\}$ by

$$\hat{F} \mid R(\eta, \kappa, \sigma) = \{F_{\sigma K_a}(\tau) \mid R(\eta, \kappa, \sigma, \sigma K_a)\}.$$

A direct result of (28) is

$$R(\vartheta, \vartheta, \vartheta^2) \cdot R(\eta, \kappa, \sigma) = R(\vartheta\eta, \vartheta\kappa, \vartheta^2\sigma) \quad \text{for} \quad \{K, -k\} \tag{35}$$

and in particular

$$R(\Theta, \Theta, \Theta^2)\, R(\vartheta, \vartheta, \vartheta^2) = R(\Theta\vartheta, \Theta\vartheta, \Theta^2\vartheta^2) \quad \text{for} \quad \{K, -k\}. \tag{36}$$

In addition, we have the following lemma:

Lemma 30: The operator $R(\vartheta, \vartheta, \vartheta^2)$ transforms form vectors of type $\{K, -k\}$ into form vectors of the same type.

Proof: The component /382

$$F_{\vartheta^2 K_a}(\tau) \mid R(\bar{\vartheta}, \vartheta, \vartheta^2, \bar{\vartheta}^2 K_a)$$

of $\hat{F} \mid R(\vartheta, \vartheta, \vartheta^2)$ which belongs to the class K_a is clearly a modular form of type $(K_a, -k)$.

Lemma 31: Let $\hat{F} \in \{K, -k, \chi\}$; then

$$\hat{F} \mid R(\vartheta, \vartheta, \vartheta^2) = N(\vartheta)^{-k} \chi(\vartheta^2) \cdot \hat{F} \quad \text{for} \quad \vartheta \in K.$$

Proof: The component of $\hat{F} \mid R(\vartheta, \vartheta, \vartheta^2)$ which belongs to the class K_a is

$$F_{K_a}(\tau) \Big| \begin{pmatrix} \vartheta & 0 \\ 0 & \vartheta \end{pmatrix} = N\,\vartheta^{-k} F_{K_a}(\tau) = N(\vartheta)^{-k} \chi(\vartheta^2) F_{K_a}(\tau).$$

From these two lemmas and from (36), it follows that the operators

$$R(\vartheta, \vartheta, \vartheta^2)\, N(\vartheta)^k \bar{\chi}(\vartheta^2) \text{ with } \vartheta \in Z^\times$$

induce a representation of the finite abelian number class group $Z^\times/K^\times$ in the family of all modular forms of type $\{K_a, -k, \chi\}$.

Let $\psi(K_a)$ be a character of the number class group, that is, let $\psi(K_a^\times)\, \psi(K_b^\times) = \psi(K_a^\times K_b^\times)$. We will also set $\psi(\sigma) = \psi(K_a^\times)$ for $\sigma \in K_a$. Let a form vector $\hat{F}$ of type $\{K, -k, \chi\}$ be called a form vector of type $\{K, -k, \chi, \psi\}$ if the following condition holds:

$$\hat{F} \mid R(\vartheta, \vartheta, \vartheta^2)\, N(\vartheta)^k \bar{\chi}(\vartheta^2) = \psi(\vartheta) \cdot \hat{F} \quad \text{for} \quad \vartheta \in Z^\times \tag{37}$$

On the basis of familiar theorems from representations theory for finite abelian groups, we know that the family $\{K, -k, \chi\}$ is the direct sum of the families $\{K, -k, \chi, \psi\}$.

We now define the T-operators for form vectors $\hat{F} = \{F_{K_a}(\tau)\}$ of type $\{K, -k, \chi\}$ by

$$\hat{F} \mid T(\sigma) = \{F_{\sigma K_a}(\tau) \mid T_{\sigma K_a}(\sigma)\}.$$

Lemma 32: The family of all form vectors of type $\{K, -k, \chi\}$ is projected onto itself by the operator $T(\sigma)$.

Proof: From Lemma 29, the component of $\hat{F}|T(\sigma)$ which belongs to the class K_a is

$$F_{\sigma K_a}(\tau) \mid T_{\sigma K_a}(\sigma) \in (K_a, -k, \chi).$$

Hence the T-operators for form vectors can be applied one after the other, and Theorem 6 follows immediately:

Lemma 33: The multiplication theorem for T-operators of form vectors of type $\{K, -k, \chi\}$ is given by

$$T(\Sigma)\, T(\sigma) = \sum_{\substack{(\vartheta) \\ 1/\vartheta/(\Sigma, \sigma)}} R(\vartheta, \vartheta, \vartheta^2)\, \mathrm{N}(\vartheta)^{2k-1}\, \bar{\chi}(\vartheta^2)\, T\left(\frac{\Sigma\sigma}{\vartheta^2}\right). \quad *$$

In particular, the T-operators for form vectors are thus commutable. If we further limit the region where the operators can be applied, (37) leads to

Theorem 7: The multiplication theorem for T-operators of form vectors of type $\{K, -k, \chi, \psi\}$ is given by

$$T(\Sigma)\, T(\sigma) = \sum_{\substack{(\vartheta) \\ 1/\vartheta/(\Sigma, \sigma)}} N(\vartheta)^{k-1}\, \psi(\vartheta)\, T\left(\frac{\Sigma\sigma}{\vartheta^2}\right).$$

This equation is now formally equivalent to Hecke's multiplication theorem for T-operators. /383

*[Translator's note: Handwritten comment appears in the original to the right of equation: Essential for Euler product.]

§7. The Fourier Expansion of a Form Vector

First, we look at a modular form $F_{K_a}(\tau) \in (K_a, -k)$. This form satisfies the transformation formula

$$F_{K_a}(\tau+\zeta) = F_{K_a}(\tau) | U^{\zeta} = F_{K_a}(\tau) \text{ for integer } \zeta \in K_a^{-1},$$

that is, $F_{K_a}(\tau)$ is a periodic function. A bit of reflection reveals that $F_{K_a}(\tau)$ possesses a Fourier series of the form

$$F_{K_a}(\tau) = \sum_{\substack{(\mu) \\ \frac{1}{\delta}/\mu}} a(\mu, \omega)\, e(\mu\tau) \qquad \text{for} \quad \tau \in \hat{T}_{\omega} \tag{38}$$

in every T, assuming $e^{2\pi i S\mu\tau} = e(\mu\tau)$. The first argument of the Fourier coefficients is the index of summation; the second denotes the region of convergence, and can thus be varied within its class. Since $F_{K_a}(\tau)$ is restricted to the region $\operatorname{Im}\frac{\omega\tau}{|\omega|} > 1$, (38) reduces to

$$F_{K_a}(\tau) = a(0, \omega) + \sum_{\substack{\mu \\ \frac{1}{\delta}/\mu \in K_a^{\times} \\ \mu \gtrapprox \omega}} a(\mu, \omega)\, e(\mu\tau) \qquad \text{for} \quad \tau \in \hat{T}_{\omega}$$

and $a(\mu, \omega)$ can be replaced by $a(\mu, \mu)$. A simple notation change yields:

Lemma 34: Let $F_{K_a}(\tau)$ be a modular form of type $(K_a, -k)$; then there exists for every ideal number $\omega \in K_a^{\times}$ a Fourier coefficient $a_0(\omega)$, and for every ideal number $\mu \in K_a^{\times}$ with $\frac{1}{\delta}/\mu$ a Fourier coefficient $a(\mu)$ such that

$$F_{K_a}(\tau) = a_0(\omega) + \sum_{\substack{\mu \\ \frac{1}{\delta}/\mu \in K_a^{\times} \\ \mu \gtrapprox \omega}} a(\mu)\, e(\mu\tau) \qquad \text{for} \quad \tau \in \hat{T}_{\omega} \tag{39}$$

given $e^{2\pi i S\mu\tau} = e(\mu\tau)$.

A direct result of Lemma 34 is

Lemma 35: Let $\hat{F}$ be a form vector of type $\{K, -k\}$; then there exists for every ideal number $\omega \in Z^{\times}$ a coefficient $a_0(\omega)$, and for every ideal number $\mu \in Z^{\times}$ with $\frac{1}{\partial} / \mu$ a coefficient $a(\mu)$ such that the component of $\hat{F}$ which belongs to the class K_a has the Fourier series (3a) in the region T_ω (with $\omega \in K_a$).

Now let $\hat{F}$ be a form vector of type $\{K, -k, \chi\}$. We will set $a_0(\omega) = c_0(\omega)\,\chi(\omega)$ and $a(\mu) = c(\sigma)\,\overline{\chi}(\mu)$, where $\sigma = \partial\mu$. $c(\sigma)$ is now defined for all integer σ. For every component of $\hat{F}$, the following holds:

$$F_{K_a}(\varepsilon\tau) = \chi(\varepsilon)\, F_{K_a}(\tau). \tag{40}$$

Substituting in the Fourier series on both sides and comparing coefficients yields /384

$$c_0(\omega) = c_0(\varepsilon\,\omega), \qquad c(\sigma) = c(\varepsilon\,\sigma);$$

that is, the coefficients $c_0(\omega)$ and $c(\sigma)$ are dependent only on (ω) and (σ) respectively. Then we get:

Theorem 8: Let $\hat{F}$ be a form vector of type $\{K, -k, \chi\}$; then there exists for every $(\omega) \in Z^{\times}$ a coefficient $c_0(\omega)$, and for every integer (σ) a coefficient $c(\sigma)$ such that the component of $\hat{F}$ which belongs to the class K_a possesses the the series expansion

$$F_{K_a}(\tau) = c_0(\omega)\,\chi(\omega) + \sum_{\substack{\mu \\ \frac{1}{\partial} / \mu \in K_a \\ \mu \approx \omega}} c(\partial\mu)\,\overline{\chi}(\mu)\, e(\mu\tau)$$

in the region $\hat{T}_\omega$.

Now let $\hat{F} \in \{K_a, -k, \chi, \Psi\}$. Theorem 8 is valid in the form given. From the expansion of the components of $\hat{F}$, the expansion of the components of $\hat{F}|T(\sigma)$ can now be computed as well. From the definition of the type $\{K, -k, \chi, \Psi\}$, and from Theorem 8, the following holds for $\tau \in T_{\eta^{-2}\omega}$:

$$F_{\eta^{-1}K_a}(\tau) = F_{K_a}(\tau) \mid R(\eta, \eta, \eta^2, K_a)\, \mathrm{N}(\eta)^k\, \bar{\chi}(\eta^2)\, \bar{\psi}(\eta)$$
$$= c_0(\eta^{-2}\omega)\, \chi(\eta^{-2}\omega) + \sum_{\substack{\mu \\ \frac{1}{\partial}/\mu \varepsilon \eta^{-1}K_a \\ \mu \approxeq \eta^{-2}\omega}} c(\partial\mu)\, \bar{\chi}(\mu)\, e(\mu\tau).$$

Applying the operator $\begin{pmatrix} 1 & 0 \\ 0 & \eta^{-2}\sigma \end{pmatrix}$ to both sides, (26) and (27) yield the following for $\tau \in T_{\sigma^{-1}\omega}$:

$$F_{K_a}(\tau) \mid R(\eta, \varkappa, \sigma, K_a)\, \mathrm{N}(\eta)^k\, \bar{\chi}(\eta^2)\, \bar{\psi}(\eta)$$
$$= \mathrm{N}\left(\frac{\sigma}{\eta^2}\right)^{-k} \chi\left(\frac{\sigma^2}{\eta^4}\right) \left\{c_0(\eta^2\omega)\, \chi(\eta^{-2}\omega) + \sum_{\mu} c(\partial\mu)\, \bar{\chi}(\mu)\, e\left(\frac{\mu\eta^2\tau}{\sigma}\right)\right\}.$$

Setting $\mu = \frac{\sigma\mu'}{\eta^2}$ yields

$$F_{K_a}(\tau) \mid R(\eta, \varkappa, \sigma, K_a)\, \mathrm{N}(\varkappa)^k\, \bar{\chi}(\sigma)\, \bar{\psi}(\eta) =$$
$$= c(\eta^{-2}\omega)\, \chi\left(\frac{\omega\sigma}{\eta^4}\right) + \sum_{\substack{\mu' \\ \frac{\eta^2}{\partial\sigma}/\mu' \in \sigma^{-1}K_a \\ \mu' \approxeq \omega\sigma^{-1}}} c\left(\frac{\partial\sigma\mu'}{\eta^2}\right) \bar{\chi}(\mu')\, e(\mu'\tau).$$

We see without further ado that $\hat{F} \in \{K, -k, \chi, \Psi\}$ is a cusp form vector if and only if all the coefficients $c_0(\omega)$ are zero. We have now proven:

Theorem 9: Let $\{c_0(\omega), c(\Sigma)\}$ be the system of Fourier coefficients for the form vector $\hat{F} \in \{K, -k, \chi, \Psi\}$; let $\{c_0^\sigma(\omega), c^\sigma(\Sigma)\}$ be the system of Fourier coefficients for the form vector $\hat{F}|T(\sigma)$. Then

$$c_0^\sigma(\omega) = \sum_{\substack{(\vartheta) \\ 1/\vartheta/\sigma}} \mathrm{N}(\vartheta)^{k-1}\, \psi(\vartheta)\, c_0\left(\frac{\sigma\omega}{\vartheta^2}\right) \chi\left(\frac{\sigma^2}{\omega^4}\right),$$
$$c^\sigma(\Sigma) = \sum_{\substack{(\vartheta) \\ 1/\vartheta/(\Sigma,\sigma)}} \mathrm{N}(\vartheta)^{k-1}\, \psi(\vartheta)\, c\left(\frac{\Sigma\sigma}{\vartheta^2}\right).$$

Proof: Compute the Fourier expansion of the component of $\hat{F}|T(\sigma)$ belonging to the class K_a and compare coefficients. All necessary transformations are permissable, since the Fourier series' converge absolutely. When carrying out /385 the computation, observe that

$$e(\mu\tau) \mid U^{\zeta} = e(\mu\zeta)\, e(\mu\tau)$$

and

$$\mathrm{N}(\varkappa)^{-1} \sum_{\substack{\zeta \\ \frac{1}{\eta} / \zeta \bmod \frac{\varkappa}{\eta} \\ \zeta \sim \mu^{-1}}} e(\mu\zeta) = \begin{cases} 1 & \text{where } \eta / \partial\mu, \\ 0 & \text{otherwise} \end{cases}$$

The coefficients $c_0(\omega)$ are determined by the remaining coefficients, as there are no constant modular forms of type $(K_a, -k)$ $(a = 1, 2, \ldots, h)$ which differ from the null form. In order to express the coefficients $c_0(\omega)$ explicitly in terms of the others, we require

Theorem 10: Let $\hat{F} = \{F_{K_a}(\tau)\} \in \{K, -k, \chi, \psi\}$; then

$$F_{K_a}\left(-\frac{1}{\tau}\right) = \mathrm{N}\,\tau^k\, \psi(K_a)\, F_{K_a^{-1}}(\tau) \qquad \text{for} \quad \tau \in \mathfrak{S}_{K_a^{-1}}.$$

Proof: We start from

$$F_{K_a}(\tau) \mid R(\vartheta, \vartheta, \vartheta^2, K_a)\, \mathrm{N}(\vartheta)^k\, \bar{\chi}(\vartheta^2)\, \bar{\psi}(\vartheta) = F_{\vartheta^{-1}K_a}(\tau).$$

letting $\vartheta \in K_a$; thus we can set $R(\vartheta, \vartheta, \vartheta^2, K_a) = \begin{pmatrix} 0 & -\vartheta \\ \vartheta & 0 \end{pmatrix}$, and we get

$$F_{K_a}\left(-\frac{1}{\tau}\right) \mathrm{N}\,\tau^{-k}\, \mathrm{N}(\vartheta)^k\, \mathrm{N}\,\vartheta^{-k}\, \bar{\chi}(\vartheta^2)\, \bar{\psi}(\vartheta) = F_{K_a^{-1}}(\tau).$$

from which Theorem 10 follows immediately.

Lemma 36: Let $\hat{F}$ be a form vector of type $\{K, -k\, \chi, \Psi\}$; let $\{c_0(\omega), c(\sigma)\}$ be the system of its Fourier coefficients. Then

$$c_0(\omega)\, \chi(\omega) = i^{nk}\, \bar{\chi}(\omega^2)\, \psi(\omega) \lim_{y \to 0} \mathrm{N}\, y^k \sum_{\substack{\mu \\ \frac{1}{\partial} / \mu \\ \mu \approx \omega^{-1}}} c(\partial\mu)\, \bar{\chi}(\mu)\, e(i\,|\mu|\, y).$$

Proof: Let $\tau = i \frac{\omega}{|\omega|} y$ with $y > 0$: Then according to Theorem 10 and Eq. (26),

$$F_{K_a}\left(i \frac{|\omega|}{\omega} y^{-1}\right) = i^{nk} \bar{\chi}(\omega^2) \psi(\omega) \mathrm{N}\, y^k F_{K_{\bar{a}}^{-1}}\left(i \frac{|\omega|}{\omega} y\right).$$

Substituting in the Fourier series' on both sides, we get

$$\begin{aligned} c_0(\omega)\chi(\omega) = & -\sum_{\mu \approx \omega}' c(\partial\mu)\bar{\chi}(\mu) e\left(\mu i \frac{|\omega|}{\omega} y^{-1}\right) \\ & + i^{nk} \bar{\chi}(\omega^2) \psi(\omega) c_0\left(\frac{1}{\omega}\right) \chi\left(\frac{1}{\omega}\right) \mathrm{N}\, y^k \\ & + i^{nk} \bar{\chi}(\omega^2) \psi(\omega) \mathrm{N}\, y^k \sum_{\mu \approx \omega^{-1}}' c(\partial\mu)\bar{\chi}(\mu) e\left(\mu i \frac{\omega}{|\omega|} y\right). \end{aligned}$$

Carrying out the limit $y \to 0$ yields the proof of Lemma 36.

Lemma 37: Every form vector of type $\{K, -k, \chi\}$ is /386
a cusp form vector if there exists a unity $\varepsilon_0 > 0$ such that $\chi(\varepsilon_0) \neq +1$. An example of such a unity is ε_1^2, where $\mathrm{N}\varepsilon_1 = -1$ and K is odd.

Proof: First let $\hat{F} \in \{K, -k, \chi, \Psi\}$, Ψ arbitrary. According to Theorem 8, the coefficient $c_0(\omega)$ is dependent only on (ω) and $a_0(\omega) = c_0(\omega)\chi(\omega)$ is dependent only on $\hat{T}_\omega$. Thus, for $\varepsilon_0 > 1$, $\chi(\varepsilon_0) \neq 1$,

$$c_0(\omega)\chi(\varepsilon_0\omega) = c_0(\varepsilon_0\omega)\chi(\varepsilon_0\omega) = c_0(\omega)\chi(\omega),$$

hence $c_0(\omega) = c_0(\omega)\chi(\varepsilon_0)$, hence $c_0(\omega) = 0$; that is, $\{K, -k, \chi, \Psi\}$ contains only cusp form vectors. Since the family $\{K, -k, \chi\}$ can be represented as the direct sum of the families $\{K, -k, \chi, \Psi\}$, the family $\{K, -k, \chi\}$ also contains only cusp form vectors.

Lemma 38: Let $\hat{F}$ be a form vector of type $\{K, -k, \chi\}$ with $k \geq 2$; let $\{c_0(\omega), c(\sigma)\}$ be the system of its Fourier coefficients. Then

$$c(\sigma) = O(\mathrm{N}(\sigma)^{k-1+\varepsilon}) \quad \text{for every} \quad \varepsilon > 0. \tag{41}$$

If $\hat{F}$ is a cusp form vector, then it is also true (even for $k = 1$) that

$$c(\sigma) = O\left(N(\sigma)^{\frac{k}{2}}\right). \tag{42}$$

Proof: We look at the function branch of the component $F_{K_a}(\tau)$ which belongs to the region T_ω, where we can assume that ω is an integer. Applying the operator $\begin{pmatrix} 1 & 1 \\ 0 & \omega \end{pmatrix}$ yields a function defined in the region $\hat{T}_1$ which is a modular form of order (ω) in the sense used by Kloosterman. Hence the assertion holds for the Fourier coefficients $c(\sigma)$ with $\sigma \approx \frac{\omega}{\partial}$. Because of the finiteness of the class number, the assertion is thus universally valid.

For the case $k = 1$, the approximation (42), which is weaker than (41), is universally valid.

§8. Representation of the T-Operators

Let $\hat{S}$ be an arbitrary family of form vectors of type $\{K, -k, \chi, \Psi\}$, which is mapped onto itself by all T-operators. Let the rank of $\hat{S}$ be r. In the family $\hat{S}$ we trace a basis

$$\{\hat{F}^1, \hat{F}^2, \ldots, \hat{F}^r\}$$

Thus for every operator $T(\sigma)$ there is a corresponding matrix $C(\sigma) = (c_{ab}(\sigma))$, the representation matrix of the operator $T(\sigma)$ with respect to the basis $\{\hat{F}^1, \hat{F}^2, \ldots, \hat{F}^r\}$, such that

$$\hat{F}^a \mid T(\sigma) = \sum_{b=1}^{r} c_{ab}(\tau)\hat{F}^b \tag{43}$$

From Theorem 7, these matrices $C(\sigma)$ satisfy the multiplication rule

$$C(\Sigma)\,C(\sigma) = \sum_{\substack{(\vartheta)\\ 1/\vartheta/(\Sigma,\sigma)}} C\left(\frac{\Sigma\sigma}{\vartheta^2}\right) \mathrm{N}(\vartheta)^{k-1}\,\psi(\vartheta). \tag{44}$$

The Fourier coefficients of the basis vectors of the family $\hat{S}$ /387
are arranged in a matrix with a finite number of rows and an infinite number of columns:

$$\left(\begin{array}{c|c} \ldots, c_0^1(\omega), \ldots & \ldots, c^1(\Sigma), \ldots \\ \ldots, c_0^2(\omega), \ldots & \ldots, c^2(\Sigma), \ldots \\ \ldots, \ldots, \ldots & \ldots, \ldots, \ldots \\ \ldots, c_0^r(\omega), \ldots & \ldots, c^r(\Sigma), \ldots \end{array}\right). \tag{45}$$

The constant elements are in the left submatrix, so this submatrix vanishes if $\hat{F}$ includes only cusp forms. The matrix (45) is of maximal rank. Since every linear relation between the rows of (45) would be transfered to the basis vectors. But the right submatrix is also of maximal rank, since Lemma 36 implies that every linear row relation of the right submatrix is transfered to the left submatrix.

Assume that the columns of the right submatrix is ordered according to increasing $N(\Sigma)$. Let B denote the smallest natural number such that the matrix

$$(c^b(\Sigma))\ \text{where}\ N(\Sigma) \leq B$$

is of maximal rank. Then it is clear that every form vector $\hat{F} \in \hat{\hat{S}}$ whose Fourier coefficients $c(\Sigma)$ vanish for $N(\Sigma) \leqq B$ also vanishes identically.

If we compute the Fourier coefficients of $\hat{F}^a | T(\sigma)$ according to Theorem 9, (43) yields

$$\sum c^a\left(\frac{\Sigma\sigma}{\vartheta^2}\right) N(\vartheta)^{k-1}\psi(\vartheta) = \sum_{b=1}^{r} c_{ab}(\sigma)\, c^b(\Sigma), \tag{46}$$

$$\sum_{\substack{(\vartheta)\\ 1/\vartheta/\sigma}}^{1/\vartheta/(\Sigma,\sigma)} c_0^a\left(\frac{\omega}{\vartheta}\cdot\frac{\sigma}{\vartheta}\right) N(\vartheta)^{k-1}\psi(\vartheta)\,\chi\left(\frac{\sigma^2}{\vartheta^2}\right) = \sum_{b=1}^{r} c_{ab}(\sigma)\, c_0^b(\omega). \tag{47}$$

The left side of (46) is symmetric in Σ and σ, hence so is the right side. This means that

$$\sum_{b=1}^{r} c_{ab}(\sigma)\, c^b(\Sigma) = \sum_{b=1}^{r} c_{ab}(\Sigma)\, c^b(\sigma). \tag{48}$$

We replace Σ row-wise with numbers $\Sigma_1, \Sigma_2, \ldots, \Sigma_3$ such that the matrix $(c^b(\Sigma_a))$ is of rank r. Then we can regard (48) as a linear system of equations for the $c_{ab}(\sigma)$'s with fixed (σ), and we can solve for $c_{ab}(\sigma)$. We get the $c_{ab}(\sigma)$'s as linear combinations of the $c_b(\sigma)$'s with coefficients which are independent of σ. Thus there exist r matrices B^b of degree r such that

$$C(\sigma) = \sum_{b=1}^{r} c^b(\sigma)\, B^b \tag{49}$$

We make the following definition:

$$C_0(\omega) = \sum_{b=1}^{r} c_0^b(\omega)\, B^b. \tag{50}$$

The elements in the a-th row and b-th column of the matrices $\{C_0(\sigma), C(\sigma)\}$ can thus be regarded as Fourier coefficients of /388 a form vector $\hat{F}_{ab}$. The family generated by the form vectors $\hat{F}_{ab}$ is contained in the family $\hat{S}$. In addition,

$$c^a(\sigma) = \sum_{b=1}^{r} c_{ab}(\sigma)\, c^b(1) \tag{51}$$

from which it follows, that the $\hat{F}_{ab}$'s are linearly equivalent with the F^a's.

A transformation of the matrices $C(\sigma)$ with a non-degenerate matrix corresponds to the transformation of the basis for the family $\hat{S}$ and vice versa. We know that a finite number of commutable matrices can be transformed to the triangular form: this means in our case that there is a basis in the family $\hat{S}$ such that all matrices $C(\sigma)$ with $N(\sigma) \leqq B$ have the triangular form. But since the corresponding elements of the matrices $C(\sigma)$ can be interpreted as Fourier coefficients of form vectors of the family $\hat{S}$, it follows that all matrices $C(\sigma)$ are triangular matrices. We have now proven:

> Lemma 39: In every family $\hat{S}$ of form vectors of type $\{K, -k, \chi, \Psi\}$ which is closed with respect to the T-operators, there exists a basis such that the representation matrices of all T-operators have the triangular form. Thus, in particular, there exists in the family $\hat{S}$ an eigenvector of all T-operators.

All multiples of an eigenvector $\hat{F}^1 \in \{K, -k, \chi, \Psi\}$ constitute a family of rank $r = 1$ which is closed with respect to the T-operators. (43) now implies that

$$\hat{F}^1 \mid T(\sigma) = c_{11}(\sigma) F^1;$$

that is, $c_{11}(\sigma)$ is the eigenvalue of $\hat{F}$ with respect to $T(\sigma)$. It follows from (51) that

$$c_{11}(\sigma)\, c^1(1) = c^1(\sigma).$$

Because $\hat{F}^1 \not\equiv 0$, it follows that $c^1(1) \neq 1$. Eigenvectors which satisfy this condition are called normalized eigenvectors.

Theorem 11: Let $\hat{F} \in \{K, -k, \chi, \Psi\}$ be a normalized eigenvector of all T-operators; let the system of Fourier coefficients of $\hat{F}$ be $\{c_0(\omega), c(\sigma)\}$. Then

$$\hat{F} \mid T(\sigma) = c(\sigma)\,\hat{F}$$

and

$$c(\Sigma)\,c(\sigma) = \sum_{\substack{(\vartheta) \\ 1/\vartheta/(\Sigma,\sigma)}} c\left(\frac{\Sigma\sigma}{\vartheta^2}\right) \mathrm{N}(\vartheta)^{k-1}\,\psi(\vartheta).$$

Proof: The first assertion has already been proven. So the second assertion follows from the multiplication rule for the T-operators.

Thus, in particular, the Fourier coefficients of a normalized eigenvector are multiplicative with respect to relatively prime arguments. The inverse is also true. The proof of this requires the following lemma:

Lemma 40: Let $\rho \in Z^{\times}$ be a given integer and $\hat{F}$ a form vector of type $\{K, -k, \chi, \Psi\}$. Let the Fourier coefficients be

$$c(\sigma) = 0 \quad \text{for the case} \quad \rho/\sigma.$$

Then $\hat{F} \equiv 0$.

Proof: It follows from the given that /389

$$F_{K_a}(\tau) \left| \begin{pmatrix} 1 & \frac{\zeta}{\varrho} \\ 0 & 1 \end{pmatrix} \right. = F_{K_a}\left(\tau + \frac{\zeta}{\varrho}\right) = F_{K_a}(\tau).$$

This can also be written as

$$F_{\varrho^2 K_a}(\tau) \mid R(\varrho, \varrho, \varrho^2, \varrho^2 K_a) \begin{pmatrix} 1 & \zeta\varrho^{-1} \\ 0 & 1 \end{pmatrix} = \mathrm{N}(\varrho)^{-k}\,\chi(\varrho^2)\,\psi(\varrho)\,F_{K_a}(\tau).$$

We now choose $\alpha \in \rho^{-1}K$, $\gamma \in \rho K_a$, $\delta \in \rho K_a^{-1}$ such that

α, γ, ζ, ρ are pairwise relatively prime. Then let

$$R(\varrho, \varrho, \varrho^2, \varrho^2 K_a) = \begin{pmatrix} \alpha\varrho & \beta\varrho \\ \gamma\varrho & \delta\varrho \end{pmatrix}.$$

We can now construct

$$V_{\varrho^2 K_a} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \varrho & \zeta \\ 0 & \varrho \end{pmatrix} = R(\varrho, \varrho, \varrho^2, \varrho^2 K_a) \begin{pmatrix} 1 & \zeta\varrho^{-1} \\ 0 & 1 \end{pmatrix}.$$

We execute this construction for all K_a and set

$$\hat{F} \mid \underset{\smile}{V} = \{F_{\varrho^2 K_a} \mid V_{\varrho^2 K_a}\}.$$

On the one hand, we have

$$\hat{F} \mid \underset{\smile}{V} = \mathrm{N}(\varrho)^{-k} \chi(\varrho^2) \psi(\varrho) \mathfrak{F}$$

and

$$\hat{F} \mid \underset{\smile}{V}^h = \mathrm{N}(\varrho)^{-hk} \chi(\varrho^{2h}) \mathfrak{F};$$

on the other hand

$$\hat{F} \mid V^h = \{F_{K_a} \mid W_{K_a}\} \text{ where } W_{K_a} = V_{K_a} V_{\varrho^{-2} K_a} \cdots V_{\varrho^{-2(h-1)} K_a}.$$

But the matrix W_{K_a} has determinant $\rho^{2h} \in K^{\times}$ and relatively prime elements. Thus there exist matrices $L_1, L_2 \in \Gamma_1(K_a)$ such that

$$L_1 W_{K_a} L_2 = \begin{pmatrix} 1 & 0 \\ 0 & \varrho^{2h} \end{pmatrix}$$

Thus it follows that

$$F_{K_a}(\tau) \mid \begin{pmatrix} 1 & 0 \\ 0 & \varrho^{2h} \end{pmatrix} = \mathrm{N}(\varrho)^{-hk} \chi(\varrho^{2h}) F_{K_a}$$

and F_{K_a} is constant.

Theorem 12: Let $\hat{F}$ be a form vector of type $\{K, -k, \chi, \Psi\}$; let $\{c_0(\omega), c(\sigma)\}$ be the system of Fourier coefficients of $\hat{F}$. Then it follows from

$$c(\Sigma)\, c(\sigma) = c(\Sigma\, \sigma) \text{ for } (\Sigma, \sigma) = 1$$

that $\hat{F}$ is an eigenvector of all T-operators.

Proof: Let ρ be a prime number. Calculation shows that all Fourier coefficients $c^*(\sigma)$ of the form vector

$$\hat{F}^* = \hat{F} \mid T(\varrho) - c(\varrho)\hat{F}$$

for which ρ / σ is valid vanish. Thus it follows, according to Lemma 40, that $\hat{F}^*$ vanishes identically. Thus $\hat{F}$ is an eigenvector of all $T(\rho)$ where ρ is a prime number. Since the $T(\sigma)$'s all lie in the operator ring generated by $T(\rho)$, $\hat{F}$ is an eigenvector of all $T(\sigma)$.

§9. The Metrization of the Form Vectors /390

Let $\hat{F} = \{F_{K_a}(\tau)\}$ and $\hat{F}^* = \{F^*_{K_a}(\tau)\}$ be two form vectors.

Using the notation of Lemma 18, the following is true for every matrix $M \in \Gamma(K_a)$:

$$F_{K_a}(\tau) \mid M = \mathrm{N}(\gamma\tau+\delta)^{-k} F_{K_a}(M\tau),$$
$$F^*_{K_a}(\tau) \mid M = \mathrm{N}(\gamma\tau+\delta)^{-k} F^*_{K_a}(M\tau),$$
$$|\sigma|\, y = |\sigma| \operatorname{Im} \frac{\omega\tau}{|\omega|} = |\gamma\tau+\delta|^2 \operatorname{Im} \frac{\omega'\tau'}{\omega'} = |\gamma\tau+\delta|^2\, y'.$$

It follows that

$$\mathrm{N}(\sigma)^k F_{K_a}(\tau) \mid M \cdot \overline{F^*_{K_a}(\tau) \mid M} \cdot \mathrm{N}\, y^k = F_{K_a}(M\tau)\, \overline{F^*_{K_a}(M\tau)}\, \mathrm{N}\, y'^k. \tag{52}$$

If we choose $M = L \in \Gamma_1(K_a)$, we get

$$F_{K_a}(\tau)\, \overline{F^*_{K_a}(\tau)}\, \mathrm{N}\, y^k = F_{K_a}(L\tau)\, \overline{F^*_{K_a}(L\tau)}\, \mathrm{N}\, y'^k. \tag{53}$$

Thus the integral

$$(F_{K_a}, F^*_{K_a})_{K_a} = \iint_{P_1(K_a)} F_{K_a}(\tau)\, \overline{F^*_{K_a}(\tau)}\, \mathrm{N}\, y^k \frac{dx\,dy}{y^2},$$

where $y = \operatorname{Im} \frac{\omega\tau}{|\omega|}$; $x = \operatorname{Re} \frac{\omega\tau}{|\omega|}$ must be entered, is independent of the choice of the fundamental domain. The existence of the integral is in any case assured if F_{K_a} or $F^*_{K_a}$ is a cusp form.

This is easily seen when one decomposes the fundamental domain $\hat{P}_1(K_a)$ into the images of the regions $\hat{P}_\omega$. The existence of the integral over every such region is proven by transforming the field of integration into the corresponding $\hat{P}_{\omega_i}$, utilizing Eq. (52); in the process, we see that a necessary and sufficient condition for existence is that at least one of the two modular forms must vanish in every cusp. We assume, for the sake of simplicity, that one of the two form vectors is a cusp vector. We now define the scalar product[3] of two form vectors, at least

[3] This scalar product is defined analogously to Petersson's scalar product of the modular forms for a rational modular group [11].

one of which is a cusp form vector, as follows:

$$(\hat{F},\hat{F}^*) = \sum_{K_a} (F_{K_a}, F^*_{K_a})_{K_a}.$$

The following lemma is immediately clear:

Lemma 41: 1. Let $\hat{F}$ or $\hat{F}^*$ be a cusp form vector. Then

$$(\hat{F},\hat{F}^*) = \overline{(\hat{F}^*,\hat{F})}$$

2. Let the r form vectors $\hat{F}_a$ or the s form vectors $\hat{F}^*_b$ be cusp form vectors; let λ_a, λ^*_b be arbitrary complex numbers ($a = 1, 2, \ldots, r$: $b = 1, 2, \ldots, s$). Then

$$\left(\sum_a \lambda_a \hat{F}_a, \sum_b \lambda^*_b \hat{F}^*_b\right) = \sum_{a,b} \lambda_a \overline{\lambda^*_b} (\hat{F}_a, \hat{F}^*_b).$$

3. Let $\hat{F}$ be a cusp form vector. Then

$$(\hat{F},\hat{F}) > 0,$$

whenever $\hat{F}$ is not the null vector.

We also have /391

Lemma 42: Let $\hat{F}$ or $\hat{F}^*$ be a cusp form vector. Then $(\hat{F}, \hat{F}^*) = 0$ if $\hat{F} \in \{K, -k, \chi\}$, $\hat{F}^* \in \{K, -k, \chi^*\}$, $\chi \neq \chi^*$, or if $\hat{F} \in \{K, -k, \chi, \psi\}$, $\hat{F}^* \in \{K, -k, \chi, \psi^*\}$, $\psi \neq \psi^*$.

Proof: In Eq. (52), set $M = R(1, 1, \varepsilon, K_a)$. It follows that

$$\chi(\varepsilon)\,\overline{\chi}^*(\varepsilon)\,(F_{K_a}(\tau), F^*_{K_a}(\tau))_{K_a} = (F_{K_a}(\tau), F^*_{K_a}(\tau))_{K_a}$$

and hence $(\hat{F}, \hat{F}^*) = 0$ for $\chi \neq \chi^*$.

The second assertion is proven in analogous fashion, setting

$$M = R(\vartheta, \vartheta, \vartheta^2, K_a)$$

We will call two form vectors $\hat{F}$, $\hat{F}^*$ orthogonal if their scalar product vanishes. The set of all form vectors of type $\{K, -k\}$. which are orthogonal to all cusp form vectors constitutes a linear family which we denote by $\{K, -k\}$. This family can be further split up into the subfamilies $\{K, -k, \chi\}_0$ and the subfamilies $\{K, -k, \chi, \psi\}_0$. The family $\{K, -k, \chi\}_0$ is mapped onto itself by all T-operators. This fact is a direct consequence of

Lemma 43: For form vectors $\hat{F}$, $\hat{F}^*$ of type $\{K, -k, \chi, \psi\}$, at least one of which is a cusp form vector, the following holds:

$$(\hat{F} \mid T(\sigma), \hat{F}^*) = \psi(\sigma)\,(\hat{F}, \hat{F}^* \mid T(\sigma)).$$

Proof: We use the representatives $R_0, R_1, \ldots, R_j$ of Lemma 17. Then

$$\begin{aligned}
&(F_{K_a} \mid T_{K_a}(\sigma), F^*_{\sigma^{-1}K_a})_{\sigma^{-1}K_a} \\
&\quad = \Big(F_{K_a} \mid \mathrm{N}(\sigma)^{k-1}\,\overline{\chi}(\sigma) \sum_{\mu=1}^{j} R_\mu, F^*_{\sigma K_a} \mid R_0\, \mathrm{N}(\sigma)^k\, \overline{\chi}(\sigma^2)\, \overline{\psi}(\sigma)\Big)_{\sigma^{-1}K_a} \\
&\quad = \Big(F_{K_a} \mid \mathrm{N}(\sigma)^{k-1}\,\overline{\chi}(\sigma) \sum_{\mu=1}^{j} R_\mu, F^*_{\sigma K_a} \mid R_0\, \mathrm{N}(\sigma)^k\, \overline{\chi}(\sigma^2)\, \overline{\psi}(\sigma)\Big)_{\sigma^{-1}K_a} \\
&\quad = \mathrm{N}(\sigma)^{2k-1}\chi(\sigma)\,\psi(\sigma)\,\Big(F_{K_a} \mid \sum_{\mu} R_\mu, F^*_{\sigma K_a} \mid R_0\Big)_{\sigma^{-1}K_a}.
\end{aligned}$$

We integrate not over a fundamental domain of $\overline{\Gamma}_1(\sigma^{-1}K_a)$, but over a fundamental domain of the subgroup $\overline{\Gamma}_1^\sigma(\sigma^{-1}K_a)$ of $\overline{\Gamma}_1(\sigma^{-1}K_a)$, where the matrix group $\Gamma_1^\sigma(\sigma^{-1}K_a)$ is defined as the set of all $L \in \Gamma_1(\sigma^{-1}K_a)$ with $L \equiv \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}(\sigma)$. Integration over the more extensive fundamental domain in the formation of scalar products will in the future be indicated by the superscript σ. The matrix group $\Gamma_1^\sigma(K_a)$ has a finite index in $\Gamma_1^\sigma(K_a)$; hence the index $g(\sigma) = (\overline{\Gamma}_1(K_a) : \overline{\Gamma}_1^\sigma(K_a))$ is also finite, and additionally is independent of K_a. Since the forms $F_{K_a} \mid R_\mu$ and $F^*_{\sigma K_a} \mid R_0$ act like modular forms with respect to the operators of the group $\Gamma_1^\sigma(K_a)$, we can interchange the summation and the /392 formation of the scalar product:

$$\begin{aligned}(F_{K_a} \mid T_{K_a}(\sigma), F^*_{\sigma^{-1}K_a})_{\sigma^{-1}K_a} &= \mathrm{N}(\sigma)^{2k-1}\,\chi(\sigma)\,\psi(\sigma)\frac{1}{g(\sigma)}\Big(\sum_\mu F_{K_a} \mid R_\mu, F^*_{\sigma K_a} \mid R_0\Big)^{\sigma}_{\sigma^{-1}K_a}\\ &= \mathrm{N}(\sigma)^{2k-1}\,\chi(\sigma)\,\psi(\sigma)\frac{1}{g(\sigma)}\sum_\mu \big(F_{K_a} \mid R_\mu, F^*_{\sigma K_a} \mid R_0\big)^{\sigma}_{\sigma^{-1}K_a}.\end{aligned}$$

Applying (52) with $R^*_\mu = M$ and observing that all functions are modular forms with respect to the underlying group for scalar product, we get

$$\begin{aligned}(F_{K_a} \mid T_{K_a}(\sigma), F^*_{\sigma^{-1}K_a})_{\sigma^{-1}K_a} &= \mathrm{N}(\sigma)^{k-1}\,\chi(\sigma)\,\psi(\sigma)\frac{1}{g(\sigma)}\sum_\mu \big(F_{K_a}, F^*_{\sigma K_a} \mid R_0 R_\mu^{-1}\big)^{\sigma}_{K_a}\\ &= \psi(\sigma)\frac{1}{g(\sigma)}\big(F_{K_a}, F^*_{\sigma K_a} \mid T_{\sigma K_a}(\sigma)\big)^{\sigma}_{K_a}.\end{aligned}$$

We have made use of Lemma 17. But now there are modular forms of the ordinary modular group on the right side of the equation, so it suffices to integrate over a fundamental domain of the ordinary modular groups, where we must again multiply with the group index. Thus we have proven

$$(F_{K_a} \mid T_{K_a}(\sigma), F^*_{\sigma^{-1}K_a})_{\sigma^{-1}K_a} = \psi(\sigma)\big(F_{K_a}, F^*_{\sigma K_a} \mid T_{\sigma K_a}(\sigma)\big)_{K_a}.$$

Lemma 44 follows when we sum over all classes K_a.

In the family $\{K, -k, \chi, \psi\}$, a unitary matrix is defined by way of the scalar product. Starting from an arbitrary basis $\{\hat{F}^1, \hat{F}^2, \ldots, \hat{F}^r\}$, we can easily assure, through a basis transformation, that all matrices $C(\sigma)$ have the triangular form, as was shown in the last section. There exists another transformation, which does not affect the triangular form of the matrices, such that the new basis vectors will be orthogonal and normalized in the direction of the metric. Hence we can assume from the start that the matrices $C(\sigma)$ are triangular matrices and that the basis is a normal basis. Then the matrices are diagonal matrices. In fact, we now have

$$c_{ab}(\sigma) = (\hat{F}_a \mid T(\sigma), \hat{F}_b) = \psi(\sigma)\,\overline{(\hat{F}_b \mid T(\sigma), \hat{F}_a)} = \psi(\sigma)\,\overline{c_{ab}(\sigma)}.$$

Furthermore, if $a > b$, then $c_{ba}(\sigma) = 0$, from which it follows that $c_{ab}(\sigma) \neq 0$. Thus at most the elements $c_{aa}(\sigma) \neq 0$. We have now proven:

> Theorem 13: The family $\{K, -k, \chi, \Psi\}$ is spanned by eigenvectors of all T-operators.

For the family $\{K, -k, \chi, \Psi\}_0$ the results of the last paragraph cannot be made more precise, since the scalar product of two vectors from that family generally does not exist. We can only state the following theorem:

> Theorem 14: The family $\{K, -k, \chi, \Psi\}_0$ has a basis such that the representation matrices $C(\sigma)$ of the operators $T(\sigma)$ are all triangular matrices. In particular, there exists in the family $\{K, -k, \chi, \Psi\}_0$ an eigenvector of all T-operators.

CHAPTER III

THE DIRICHLET SERIES OF A FORM VECTOR /393

§10. The Formulation of the Dirichlet Series

So that the ensuing discussion will not be complicated by too many general observations, most of the latter are collected here at the beginning of the section

Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r$ be a fixed system of basic unities of the field K; specifically, $r = n-1$. Let ϱ traverse the numbers 1, 2, ..., r and r the numbers 1, 2, ..., n, even when this is not explicitly stated. log z always denotes the principal value of the natural logarithm, which is real for positive real z. The matrix

$$\begin{pmatrix} 1 & \log|\varepsilon_1^{(1)}| & \ldots & \log|\varepsilon_r^{(1)}| \\ 1 & \log|\varepsilon_1^{(2)}| & \ldots & \log|\varepsilon_r^{(2)}| \\ \cdot & \cdots & \cdots & \cdots \\ 1 & \log|\varepsilon_1^{(n)}| & \ldots & \log|\varepsilon_r^{(n)}| \end{pmatrix} \tag{54}$$

has an inverse because the absolute value of the determinant

$$\Delta(1, \log|\varepsilon_1|, \ldots, \log|\varepsilon_r|)$$

of (54) is nR, where R is the regulator of the field K. Let the transpose of the inverse matrix for (54) be

$$\begin{pmatrix} e_0^{(1)} & e_1^{(1)} & \ldots & e_r^{(1)} \\ e_0^{(2)} & e_1^{(2)} & \ldots & e_r^{(2)} \\ \cdot & \cdot & \cdots & \\ e_0^{(n)} & e_1^{(n)} & \ldots & e_r^{(n)} \end{pmatrix}. \tag{55}$$

Clearly, $e_0^{(\nu)} = \frac{1}{n}$ ($\nu = 1, 2, \ldots, n$). The remaining elements of (55) cannot be found so easily. We must use the equations

$$\begin{gathered} S\, e_\varrho = 0, \qquad S\, e_\varrho \log|e_{\varrho'}| = \delta_{\varrho\varrho'}, \\ \frac{1}{n} + \sum_\varrho e_\varrho^{(\nu)} \log|\varepsilon_\varrho^{(\nu')}| = \delta_{\nu\nu'}, \end{gathered} \tag{56}$$

where $\delta_{\nu\nu'}$ denotes the Kronecker delta. These equations prove that (55) is the transpose of the reciprocals of (54).

With the quantities l_e, we define the characters for numbers $\mu \in Z^{\times}$ as follows:

$$\lambda(\mu) = \exp\left(\frac{\pi i}{2} \sum_{\varrho} m_{\varrho}\, \mathrm{S}\, e_{\varrho} \log|\mu|\right),$$

where $m_1, m_2, \ldots, m_r$ are fixed natural numbers. If $\lambda(\varepsilon) = 1$ for every unity ε, let λ be called a character of the first order. A character which is not of the first order but which satisfies the condition $\lambda(\varepsilon_0) = 1$ for every totally positive unity ε_0 we call a character of the second order. The remaining characters are called of the third order. Clearly a character is of the fitst order if and only if all the constants $m_1, m_2, \ldots, m_r$ are divisible by 4. The characters of the first order are those defined by Hecke [10].

Let u be a positive variable and let $x_1, \ldots, x_r$ be real variables. Next let /394

$$y = \sqrt[n]{u} \exp\left(4 \sum_{\varrho} x_{\varrho} \log|\varepsilon_{\varrho}|\right), \tag{57}$$

or more explicitly

$$y^{(\nu)} = \sqrt[n]{u} \exp\left(4 \sum_{\varrho} x_{\varrho} \log|\varepsilon_{\varrho}^{(\nu)}|\right).$$

Every n-tuple [sic] $(u; x_1, x_2, \ldots, x_r)$ is thus assigned a number y with conjugates $y^{(\nu)}$ $(\nu = 1, 2, \ldots, n)$. Equation (57) is easily solved for $(u; x_1, x_2, \ldots, x_r)$; we obtain

$$u = \mathrm{N}\, y, \qquad x_{\varrho} = \frac{1}{4} \mathrm{S}\, e_{\varrho} \log y. \tag{58}$$

The functional determinant of (57) is

$$\frac{\partial(y^{(1)}, y^{(2)}, \ldots y^{(n)})}{\partial(u, x_1, \ldots x_r)} = \Delta\left(\frac{y}{nu}, y \cdot 4\log|\varepsilon_1|, \ldots, y \cdot 4 \log|\varepsilon_r|\right) = \pm 4^r R.$$

Hence

$$\left|\frac{\partial(u, x_1, \ldots, x_r)}{\partial(y^{(1)}, y^{(2)}, \ldots, y^{(r)})}\right| = \frac{1}{4^r R}.$$

(57) and (58) can be analytically continued to the complex field. A simple substitution leads to

Lemma 44: Let u; $x_1, \ldots, x_r$ be complex variables from the region $|\arg u| < c_4$; $|\mathrm{Im}\, x_\rho| < c_5$ where c_4, c_5 are two sufficiently small positive constants. Then:

$$\tau = \frac{i|\omega|}{\omega} \sqrt[n]{u} \quad \exp\left(4 \sum_{\varrho} x_\varrho \log |\varepsilon_\varrho|\right)$$

can be solved for $(u; x_1, x_2, \ldots, x_r)$:

$$u = \mathrm{N}\frac{\omega\tau}{|\omega| i}, \; x_\varrho = \frac{1}{4} \mathrm{S}\, e_\varrho \log \frac{\omega\tau}{|\omega| i}.$$

The limitation on the u; $x_1, \ldots, x_r$ used in Lemma 39 was stipulated only for the purpose of avoiding difficulties in the selection of the function branches of the logarithm and the n-th root.

Now let $$F(\tau) = \sum_{\substack{\frac{1}{\partial}/\mu \\ \mu \approx \omega}} c(\partial\mu)\, \bar{\chi}(\mu)\, e(\mu\tau) \qquad \text{for } \tau \in \hat{T}_\omega$$

be a Fourier series with no constant elements which converges in some $\hat{T}_\omega$. Let the coefficients $c(\sigma)$ depend only on (σ); let χ be an admissable character and let

$$c(\sigma) = O(\mathrm{N}(\sigma)^\kappa).$$

If we set

$$f(u; x_1, x_2, \ldots, x_r) = F(\tau) \text{ where } \tau = \frac{i|\omega|}{\omega} \sqrt[n]{u} \quad \exp\left(4 \sum_{\varrho} x_\varrho \log \varepsilon_\varrho\right).$$

then $f(u; x_1, \ldots, x_r)$ is an ordinary periodic function in $x_1, \ldots, x_r$, so long as we choose u constant in the angular space $|\arg u| < c_4$. Since the function $f(u; x_1, \ldots, x_r)$ is analytic /395 in the variables $x_1, \ldots, x_r$ for fixed u, it is represented by its Fourier series. The Fourier coefficients are then dependent only on u, and they can be computed by the Euler-Fourier integrals /395

$$f(u;\lambda) = \int_{-\frac{1}{2}}^{+\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{+\frac{1}{2}} f(u; x_1, \ldots, x_r)\, e^{-2\pi i \sum_{\varrho} x_\varrho m_\varrho}\, dx_1 \ldots dx_r .$$

On the left side, we have chosen the character λ defined by the n-tuple $m_1, m_2, \ldots, m_r$ as the index of the Fourier coefficients. The advantage of this notation is that the Fourier coefficients are no longer dependent on the choice of basis for the unity group. We can now apply the Mellin transformation to $f(u; \lambda)$. The condition of Theorem I in reference 12 are satisfied by the functions $f(u; \lambda)$ in the angular space $|\arg u| < c_4$, so that

$$G(s;\lambda) = \int_0^\infty f(u;\lambda)\, u^{s-1}\, du \quad \text{with } \operatorname{Re} s > K+1$$

leads to the inverse relation

$$f(u;\lambda) = \frac{1}{2\pi i} \int_{(a)} G(s;\lambda)\, u^{-s}\, ds \qquad \text{with } |\arg u| < c_4,\ a > K+1$$

Thus the functions $G(s, \lambda)$ are uniquely and reversibly assigned to the function F.

We now compute the functions $G(s, \lambda)$ explictly:

$$\begin{aligned} G(s;\lambda) = \int_0^\infty \int_{-\frac{1}{2}}^{+\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{+\frac{1}{2}} \sum_{\mu} c(\partial\mu)\, \bar{\chi}(\mu) \\ \times \exp\left(-2\pi \sqrt[n]{u}\, S\, |\mu| \exp\left(4 \sum_{\varrho} x_\varrho \log |\varepsilon_\varrho|\right)\right) \\ \times e^{-2\pi i \sum x_\varrho m_\varrho}\, u^{s-1}\, dx_1, \ldots dx_r\, du . \end{aligned}$$

Summation can be substituted for integration here because of the absolute convergence of the infinite series. Furthermore, summation can be substituted for integration over u, as can be proven by the application of a familiar lemma. In the equation

$$G(s;\lambda)=\sum_{\mu} c(\partial\mu)\,\bar{\chi}(\mu)\times$$
$$\times\int_0^\infty\int_{-\frac12}^{+\frac12}\dots\int_{-\frac12}^{+\frac12}\exp\left(-2\pi\sqrt[n]{u}\,\mathrm{S}\,|\mu|\exp\left(4\sum_{\varrho}x_\varrho\log|\varepsilon_\varrho|\right)\right)e^{-2\pi i\Sigma x_\varrho m_\varrho}\times$$
$$\times u^{s-1}\,dx_1\dots dx_r\,du$$

we can arrange the series any way we like because of the absolute convergence. We set $\mu = \mu_0 \varepsilon_0 \varepsilon^4$ and sum first over all unities ε, then over a complete system of mod $\langle\varepsilon\rangle^4$ different totally positive unities ε_0, and finally over a complete system of non-associated numbers μ_0 which satisfies the initial conditions of summation. /396

Then summation over ε can be replaced by integration over the full $(x_1, x_2, \dots, x_r)$ - space, and we get

$$G(s;\lambda)=\sum_{(\mu_0)}\sum_{\varepsilon_0} c(\partial\mu)\,\bar{\chi}(\mu_0\varepsilon_0)\times$$
$$\int_0^\infty\int_{-\frac12}^{+\frac12}\dots\int_{-\frac12}^{+\frac12}\exp\left(-2\pi\sqrt[n]{u}\,\mathrm{S}\,|\mu_0\varepsilon_0|\exp\left(4\sum_{\varrho}x_\varrho\log|\varepsilon_\varrho|\right)\right)e^{-2\pi i\Sigma x_\varrho m_\varrho}\times$$
$$u^{s-1}\,dx_1\dots dx_r.$$

Now set

$$y=\sqrt[n]{u}\exp\left(4\sum_{\varrho}x_\varrho\log|\varepsilon_\varrho|\right)$$

and change the integral into an integral over the corresponding y-region. In the process, the integral decomposes into a product of n integrals. This product can be written as the norm of an integral:

$$G(s,\lambda)=\sum_{(\mu_0)}\sum_{\varepsilon_0} c(\partial\mu)\,\bar{\chi}(\mu_0\varepsilon_0)\frac{1}{4^r R}\times$$
$$\mathrm{N}\int_0^\infty e^{-2\pi|\mu_0\varepsilon_0|y}\,e^{\left(s-\frac{\pi i}{2}\sum_{\varrho}m_\varrho e_\varrho\right)\log y}\,\frac{dy}{y}$$
$$=\sum_{(\mu_0)}\sum_{\varepsilon_0} c(\partial\mu)\,\bar{\chi}(\mu_0\varepsilon_0)\frac{1}{4^r R}\times$$
$$\mathrm{N}\,e^{-\left(s-\frac{\pi i}{2}\Sigma m_\varrho e_\varrho\right)\log 2\pi|\mu_0\varepsilon_0|}\,\mathrm{N}\,\Gamma\left(s-\frac{\pi i}{2}\sum_{\varrho}m_\varrho e_\varrho\right)$$
$$=\sum_{(\mu_0)}\sum_{\varepsilon_0} c(\partial\mu)\,\bar{\chi}(\mu_0\varepsilon_0)\frac{1}{4^r R}\times$$
$$(2\pi)^{-ns}\,\mathrm{N}(\mu_0)^{-s}\,\lambda(\mu_0\varepsilon_0)\,\mathrm{N}\,\Gamma\left(s-\frac{\pi i}{2}\sum_{\varrho}m_\varrho e_\varrho\right).$$

Introducing the abbreviation

$$\Gamma(s,\lambda)=\mathrm{N}\,\Gamma\left(s-\frac{\pi i}{2}\sum_{\varrho}m_\varrho e_\varrho\right);$$

we get

$$G(s,\lambda) = \frac{1}{4^r R}(2\pi)^{-ns}\Gamma(s,\lambda)\sum_{(\mu_0)} c(\mu_0\lambda)\{\sum_{\varepsilon_0}\bar{\chi}(\varepsilon_0)\lambda(\varepsilon_0)\}\chi(\mu_0)\lambda(\mu_0)\,\mathrm{N}(\mu_0)^{-s}.$$

The expression in braces is non-zero if and only if

$$\chi(\varepsilon_0) = \lambda(\varepsilon_0) \qquad \text{for all} \quad \varepsilon_0 > 0$$

In this case, we say that χ and λ are related. Thus if χ and λ are not related, then

$$G(s,\lambda) = 0.$$

The λ's which are related to an admissible χ constitute a residue class of the group of all λ's according to the group of all λ's of the first and second class.

In the ensuing discussion, we assume that χ and λ are related, and we set

$$\bar{\chi}\lambda(\mu) = \bar{\chi}(\mu)\,\lambda(\mu).$$

Then /397

$$G(s,\lambda) = \frac{(\varepsilon_0:\varepsilon^4)}{4^r R}\Gamma(s,\lambda)(2\pi)^{-ns}\sum_{(\mu_0)} c(\partial\mu_0)\,\bar{\chi}\lambda(\mu_0)\,\mathrm{N}(\mu_0)^{-s}$$
$$= \frac{(\varepsilon_0:\varepsilon^4)}{4^r R}\bar{\chi}\lambda(\partial^{-1})\left(\frac{d}{2^n\pi^n}\right)^s\Gamma(s,\lambda)\sum_{(\mu_0)} c(\partial\mu_0)\,\bar{\chi}\lambda(\partial\mu_0)\,\mathrm{N}(\partial\mu_0)^{-s}.$$

We set $\sigma = \partial\mu_0$ and

$$D(s,\lambda) = \sum_{\substack{(\sigma)\\ \sigma\approx\partial\omega}} c(\sigma)\,\bar{\chi}\lambda(\sigma)\,\mathrm{N}(\sigma)^{-s}$$

and we get

$$G(s,\lambda) = \frac{(\varepsilon_0:\varepsilon^4)}{4^r R}\left(\frac{d}{2^n\pi^n}\right)^s\Gamma(s,\lambda)\,\bar{\chi}\lambda(\partial^{-1})\,D(s,\lambda).$$

The totality of the Dirichlet series $D(s,\lambda)$ uniquely determines the function F, as each individual step is reversible.

We collect the results of this section in

Lemma 45: Let

$$F(\tau) = \sum_{\mu\approxeq\omega} c(\partial\mu)\,\bar{\chi}(\mu)\,e(\mu\tau)$$

be a Fourier series with no constant elements. Let the coefficients $c(\sigma)$ be dependent only on (σ). Then the system of Dirichlet series

$$D(s, \lambda) = \bar{\chi}\lambda(\partial^{-1}) \sum_{\substack{(\sigma) \\ \sigma \approx \partial\omega}} c(\sigma)\,\bar{\chi}\lambda(\sigma)$$

where the character $\bar{\chi}\lambda(\sigma)$ is defined by

$$\bar{\chi}\lambda(\sigma) = \bar{\chi}(\sigma)\,\lambda(\sigma)$$

and where χ and λ are related, is uniquely and reversibly assigned to the function $F(\tau)$. The functions

$$G(s, \lambda) = \frac{(\varepsilon_0 : \varepsilon^4)}{4^r R}\left(\frac{d}{2^n \pi^n}\right)^s \Gamma(s, \lambda)\, D_\omega(s, \lambda)$$

can be expressed as integrals:

$$G(s, \lambda) = \int_0^\infty \int_{-\frac{1}{2}}^{+\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{+\frac{1}{2}} F\left(\frac{|\omega| \sqrt[n]{u}\, i}{\omega} \exp\left(4 \sum_{e} x_e \log |\varepsilon_e|\right)\right) \times$$
$$\times\, e^{-2\pi i \Sigma m_e x_e} u^{s-1}\, dx_1 \ldots dx_r\, du\,.$$

§11. The Dirichlet Series of a Form Vector

Let $\hat{F}$ be a form vector of type $\{K_a, -k, \chi, \Psi\}$. This vector is uniquely and reversibly determined by the $2^n b$ function branches

$$F_{K_a}(\tau) - c_0(\omega)\,\chi(\omega) \qquad \text{for } \bar{\tau} \in \hat{T}_\omega \tag{59}$$

The theory of the last section can be applied to the function branches, yielding

Lemma 46: Let $\hat{F}$ be a form vector of type $\{K, -k, \chi, \Psi\}$. The Dirichlet series

$$D_\omega(s;\lambda) = \bar{\chi}\lambda(\partial^{-1}) \sum_{\substack{(\sigma)\\ \sigma \gtrapprox \partial\omega}} c(\sigma)\,\bar{\chi}\lambda(\sigma)\,\mathrm{N}(\sigma)^{-s}. \tag{60}$$

correspond uniquely and reversibly to this form vector $\hat{F}$. The functions /398

$$G_\omega(s,\lambda) = \frac{(\varepsilon_0:\varepsilon^4)}{4^r R}\left(\frac{d}{2^n \pi^n}\right)^s \Gamma(s,\lambda)\, D_\omega(s,\lambda)$$

can be represented as an integral

$$G_\omega(s;\lambda) = \int_0^\infty \int_{-\frac{1}{2}}^{+\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{+\frac{1}{2}} \left\{F_{K_a}\left(\frac{|\omega|\sqrt[n]{u}\,i}{\omega}\exp(4\sum_\varrho x_\varrho \log|\varepsilon_\varrho|\right) - c_0(\omega)\,\chi(\omega)\right\} \times e^{-2\pi i \sum_\varrho m_\varrho x_\varrho} u^{s-1}\,dx_1 \ldots dx_r\,du. \tag{61}$$

The Dirichlet series converge in the half plane Re s $> \ddot{x}$ for the case $K \geqq 2$ and in the half plane Re s $> \frac{k}{2} + 1$ for the case where $\hat{F}$ is a cusp form vector. They also have a half plane of convergence for the case k = 1.

From the integral representation of the functions $G_\omega(s, \lambda)$, we see that the Dirichlet series $D_\omega^\Lambda(s, \lambda)$ are dependent in the more narrow sense only on the class of ω.

Theorem 15: The Dirichlet series $D_\omega(s, \lambda)$ can be analytically continued into the entire plane. They define integral functions so long as $\lambda \neq 1$ or $\hat{F}$ is a cusp form vector. In the special case where neither of these two conditions is satisfied, a pole of the first order is to be found at s = k. The analytic functions is defined by the Dirichlet series satisfy the functional equation

$$G_{\omega^{-1}}(k-s,\bar{\lambda}) = i^{nk}\,\chi(\omega^2)\,\psi(\omega)\,G_{\omega}(s,\lambda)\,.$$

Proof: Theorem 10 can also be written in the form

$$\begin{aligned} &F_{K_a}\left(i\frac{|\omega|}{\omega}\sqrt[n]{u}\exp\left(4\sum_{\varrho}x_{\varrho}\log|\varepsilon_{\varrho}|\right)\right)-c_0(\omega)\,\chi(\omega) \\ &\quad = i^{nk}\,\bar{\chi}(\omega^2)\,\bar{\psi}(\omega)\,u^{-k}\times \\ &\quad\times\left\{F_{K_a^{-1}}\left(i\frac{|\omega^{-1}|}{\omega^{-1}}\sqrt[n]{u^{-1}}\exp\left(-4\sum_{\varrho}x_{\varrho}\log|\varepsilon_{\varrho}|\right)\right)-c_0\left(\frac{1}{\omega}\right)\chi\left(\frac{1}{\omega}\right)\right\}+ \\ &\quad + i^{nk}\,\bar{\chi}(\omega^2)\,\bar{\psi}(\omega)\,u^{-k}\,c_0\left(\frac{1}{\omega}\right)\chi\left(\frac{1}{\omega}\right)-c_0(\omega)\,\chi(\omega) \end{aligned} \tag{62}$$

We analyze (61) at u = 1 and apply (62) to the finite partial integral. We obtain

$$\begin{aligned} G_{\omega}(s,\lambda) &= \int_1^{\infty}\int_{-\frac{1}{2}}^{+\frac{1}{2}}\cdots\int_{-\frac{1}{2}}^{+\frac{1}{2}}\left\{F_{K_a}\left(i\frac{|\omega|}{\omega}\sqrt[n]{u}\exp\left(4\sum_{\varrho}x_{\varrho}\log|\varepsilon_{\varrho}|\right)\right)-c_0(\omega)\,\chi(\omega)\right\}\times \\ &\quad\times e^{-2\pi i\sum_{\varrho}m_{\varrho}x_{\varrho}}\,u^{s-1}\,dx_1\ldots dx_r\,du\,+ \\ &\quad + i^{nk}\,\bar{\chi}(\omega^2)\,\bar{\psi}(\omega)\int_0^{1}\int_{-\frac{1}{2}}^{+\frac{1}{2}}\cdots\int_{-\frac{1}{2}}^{+\frac{1}{2}}\times \\ &\quad\times\left\{F_{K_a^{-1}}\left(i\frac{|\omega^{-1}|}{\omega^{-1}}\sqrt[n]{u^{-1}}\exp\left(-4\sum_{\varrho}x_{\varrho}\log|\varepsilon_{\varrho}|\right)\right)-c_0\left(\frac{1}{\omega}\right)\chi\left(\frac{1}{\omega}\right)\right\}\times \\ &\quad\times e^{-2\pi i\sum_{\varrho}m_{\varrho}x_{\varrho}}\,u^{s-k-1}\,dx_1\ldots dx_r\,du\,- \\ &\quad -\delta(\lambda)\left\{i^{nk}\,\bar{\chi}(\omega^2)\,\bar{\psi}(\omega)\,c_0\left(\frac{1}{\omega}\right)\chi\left(\frac{1}{\omega}\right)\frac{1}{k-s}+c_0(\omega)\,\chi(\omega)\,\frac{1}{s}\right\} \end{aligned} \tag{63}$$

$$\text{with}\quad \delta(\lambda)=\int_{-\frac{1}{2}}^{+\frac{1}{2}}\cdots\int_{-\frac{1}{2}}^{+\frac{1}{2}}e^{-2\pi i\sum_{\varrho}m_{\varrho}x_{\varrho}}\,dx_1\ldots dx_r=\begin{cases}1 & \text{where }\ \lambda=1\\ 0 & \text{otherwise}\end{cases}$$

We have extracted a summand from the finite integral and integrated it separately. The existence of the summand implies the existence of the remaining integral. If we substitute /399

$$u\to u^{-1},\ x_{\varrho}\to -x_{\varrho},$$

into the finite partial integral, we get

$$G_\omega(s,\lambda) = \int_1^\infty \int_{-\frac12}^{+\frac12} \cdots \int_{-\frac12}^{+\frac12} \left\{F_{K_a}\left(i\frac{|\omega|}{\omega}\sqrt[n]{u}\exp\left(4\sum_\varrho x_\varrho \log|\varepsilon_\varrho|\right)\right) - c_0\right\} \times$$
$$\times e^{-2\pi i \sum_\varrho m_\varrho x_\varrho} u^{s-1}\, dx_1 \ldots dx_r\, du +$$
$$+ i^{nk}\,\bar\chi(\omega^2)\,\bar\psi(\omega) \int_1^\infty \int_{-\frac12}^{+\frac12} \cdots \int_{-\frac12}^{+\frac12} \times$$
$$\times \left\{F_{K_a}^{-1}\left(i\frac{|\omega^{-1}|}{\omega^{-1}}\sqrt[n]{u}\exp\left(4\sum_\varrho x_\varrho \log|\varepsilon_\varrho|\right)\right) - c_0\left(\frac{1}{\omega}\right)\chi\left(\frac{1}{\omega}\right)\right\} \times$$
$$\times e^{-2\pi i \sum_\varrho (-m_\varrho) x_\varrho} u^{k-s-1}\, dx_1 \ldots dx_r\, du -$$
$$- \delta(\lambda)\left\{i^{nk}\,\bar\chi(\omega^2)\,\bar\psi(\omega)\, c_0\left(\frac{1}{\omega}\right)\chi\left(\frac{1}{\omega}\right)\frac{1}{k-s} + c_0(\omega)\,\chi(\omega)\frac{1}{s}\right\}.$$

The integrals are integral functions of s. Thus $G_\omega(s, \lambda)$ is analytically continued into the entire plane. $G_\omega(s, \lambda)$ is an integral function so long as $\lambda \neq 1$ or $\hat{F}$ is a cusp form vector. For the special case where neither of these conditions hold, there is a pole of the first order at $s = 0$ and $s = k$. The functional equation of the functions $G_\omega(s, \lambda)$ is obvious. Since the Γ-functions are regular functions with no zeroes, the Dirichlet series also define integer functions, aside from the Dirichlet series $D_\omega(s, 1)$. The latter may have a pole of order 1 at $s = k$. The pole of the function $G_\omega(s, 1)$ at $s = 0$ stems from a pole of order n of the Γ-function; thus the Dirichlet series $D_\omega(s, 1)$ has a zero of order at least n-1 / at s = 0.

Let $\tilde{\tilde\psi}$ be a character of the most restricted number class group, that is, let

$$\tilde{\tilde\psi}(\Sigma)\,\tilde{\tilde\psi}(\sigma) = \tilde{\tilde\psi}(\Sigma\,\sigma), \quad \psi(\sigma) = 1 \qquad \text{for} \quad \sigma \approx 1.$$

We set

$$\Lambda(\sigma) = \tilde{\tilde\psi}(\sigma)\,\bar\chi\,\lambda(\sigma),$$

for the case where $\tilde{\tilde\psi}(\sigma)\,\bar\chi\lambda(\sigma)$ is dependent only on (σ). The Dirichlet series /400

$$D(s,\Lambda) = \sum_{(\sigma)} c(\sigma)\,\Lambda(\sigma)\,\mathrm{N}(\sigma)^{-s} \tag{64}$$

are then linearly equivalent to those of Eq. (48). Since we are summing over a complete system of non-associated numbers in (64), the Dirichlet series (64) possess an Euler product expansion if the coefficients are multiplicative functions with respect to relatively prime arguments.

The following theorem follows directly from Theorem 11 and Theorem 12:

> Theorem 16: The Dirichlet series $D(s, \Lambda)$ of a form vector F of type $\{K, -k, \chi, \Psi\}$ possesses an Euler product expansion if and only if $\hat{F}$ is a normalized eigenvector of all T-operators.

A simple calculation yields the explicit form of the Euler product expansion:

$$D(s, \Lambda) = \prod_{(\varrho)} (1 - c(\varrho)\,\Lambda(\varrho)\,N(\varrho)^{-s} + \psi(\varrho)\,N(\varrho)^{k-1-2s})^{-1},$$

where the product extends over a complete system of non-associated prime numbers ρ. Thus we see that the Euler products are of the canonical form in Hecke's sense.

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